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Uno's invariant conjecture for Chevalley groups $G_2(q)$ in nondefining characteristics

Jianbei An

Department of Mathematics, University of Auckland, Auckland, New Zealand

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Abstract

Uno's invariant conjecture for the Chevalley groups $G_2(q)$ has been verified when the characteristic of the modular representation is distinct from the defining characteristic of the groups. Thus Dade's reductive conjecture and the Isaacs–Navarro conjecture both hold for $G_2(q)$ in non-defining characteristics.

Keywords: Modular representation; Blocks; Uno's conjecture; Chevalley groups

1. Introduction

Isaacs and Navarro recently in [22] proposed a new significant conjecture, and Uno [27] raised an alternating sum version of the Isaacs–Navarro conjecture which is a refinement of Dade's conjecture. In this paper, we prove Uno's invariant conjecture for $G_2(q)$ when the characteristic r of the modular representation is distinct from the characteristic of the group. Thus the Dade invariant and Isaacs–Navarro conjectures both hold for $G_2(q)$ in the non-defining characteristics. If $q \neq 3$ and 4, then Dade's reductive conjecture is equivalent to his invariant one; if $q = 3$ or 4, then the reductive conjecture is equivalent to projective invariant conjecture, which is verified by Shih-chang Huang.

The paper is organized as follows. In Section 2, we fix notation, state the Isaacs–Navarro conjecture, the Dade and Uno invariant conjectures in detail. In Section 3, we determine the stabilizers in the automorphism group $\mathrm{Aut}(G_2(q))$ of radical chains of the $G_2(q)$-invariant subfamily

E-mail address: an@math.auckland.ac.nz.

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$\mathcal{R}^0(G_2(q))$ given in [2], using the subgroup and geometry structures given by Aschbacher [5]. In Sections 4 and 5, we prove Uno's invariant conjecture for $G_2(q)$ when r is odd and even, respectively.

2. The conjectures

Let G be a finite group and r a prime. Given an r-subgroup chain

$$C\colon\ P_0 < P_1 < \cdots < P_n \tag{2.1}$$

of G, define $|C| = n$, $C_k\colon\ P_0 < P_1 < \cdots < P_k$, $C^k\colon\ O_p(G) < P_{k+1} < \cdots < P_n$ and

$$N(C) = N_G(C) = N(P_0) \cap N(P_1) \cap \cdots \cap N(P_n). \tag{2.2}$$

Then C is said to be *radical* if it satisfies the following two conditions:

(a) $P_0 = O_r(G)$ and
(b) $P_k = O_r(N(C_k))$ for $1 \leqslant k \leqslant n$,

where $O_r(N(C_k))$ is the largest normal r-subgroup of $N(C_k)$. Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical r-chains of G, $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G and $\mathrm{Blk}(G)$ the set of r-blocks.

Let E be an extension of G, $F = E/G$, $C \in \mathcal{R}(G)$, $\psi \in \mathrm{Irr}(N_G(C))$ and let $N_E(C, \psi)$ be the stabilizer of (C, ψ) in E. Then $N_F(C, \psi) = N_E(C, \psi)/N_G(C)$ is a subgroup of F. For a subgroup $U \leqslant F$, denote by $\mathrm{Irr}(N_G(C), B, d, U)$ the set of characters ψ in $\mathrm{Irr}(N_G(C))$ such that $d(\psi) = d$, $B(\psi)^G = B$ and $N_F(C, \psi) = U$, where $d(\psi) = \log_r(|G|_r) - \log_r(\psi(1)_r)$ is the r-defect of ψ and $B(\psi)$ is the block of $N_G(C)$ containing ψ.

Let $H \leqslant G$, $\varphi \in \mathrm{Irr}(H)$ and let $w(\varphi) = w_r(\varphi)$ be the integer $0 < w(\varphi) \leqslant (r-1)$ such that the r'-part $(|H|/\varphi(1))_{r'}$ of $|H|/\varphi(1)$ satisfies

$$w(\varphi) \equiv \left(\frac{|H|}{\varphi(1)}\right)_{r'} \pmod r.$$

Given an integer $w \geqslant 1$, let $\mathrm{Irr}(H, [w])$ be the subset of $\mathrm{Irr}(H)$ consisting of characters φ such that $w(\varphi) \equiv \pm w \pmod r$, and let $\mathrm{Irr}(H, B, d, U, [w]) = \mathrm{Irr}(H, B, d, U) \cap \mathrm{Irr}(H, [w])$ and $\mathrm{k}(H, B, d, U, [w]) = |\mathrm{Irr}(H, B, d, U, [w])|$.

Let $B \in \mathrm{Blk}(G)$ with a defect group $D = D(B)$ and the Brauer correspondent $b \in \mathrm{Blk}(N_G(D))$. Then

$$\mathrm{k}\big(N_G(D), B, d(B), [w]\big) = \sum_{U \leqslant F} \mathrm{k}\big(N_G(D), B, d(B), U, [w]\big)$$

is the number of characters $\varphi \in \mathrm{Irr}(b)$ such that φ has height 0 and $w(\varphi) \equiv \pm w \pmod r$, where $d(B)$ is the defect of B.

The Isaacs–Navarro Conjecture. *(See [22, Conjecture B].) In the notation above,*

$$\mathrm{k}\big(G, B, d(B), [w]\big) = \mathrm{k}\big(N_G(D), B, d(B), [w]\big).$$

Uno's Invariant Conjecture. *(See [27, Conjecture 3.2].) If $O_p(G) = 1$ and if $D(B) > 1$, then for any integers $d \geqslant 0$ and $1 \leqslant w < (r+1)/2$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}\big(N_G(C), B, d, U, [w]\big) = 0, \tag{2.3}$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

Set $\mathrm{k}(N_G(C), B, d, U) = |\operatorname{Irr}(N_G(C), B, d, U)|$. In the notation above Dade's invariant conjecture is stated as follows.

Dade's Invariant Conjecture. *(See [11].) If $O_p(G) = 1$ and $D(B) > 1$, then for any integer $d \geqslant 0$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}\big(N_G(C), B, d, U\big) = 0.$$

Thus if $r = 2$ or 3, then Uno's conjecture is equivalent to Dade's. If $D(B)$ is cyclic, then by [4, Proposition 2.1] and [22, Theorem 2.1], Uno's invariant conjecture and the Isaacs–Navarro conjecture both hold for B. Thus we may suppose $D(B)$ is non-cyclic.

If $F = E/G$ is cyclic, then U is determined uniquely by its order $|U|$, so we set

$$\mathrm{k}\big(N_G(C), B, d, |U|, [w]\big) = \mathrm{k}\big(N_G(C), B, d, U, [w]\big).$$

3. Stabilizers of radical chains

Let $q = p^e$ be a power of a prime p distinct from r and $\mathbb{F}_q$ the field of q elements. We shall follow the notation of [5]. In particular, V is a 7-dimensional linear space over $\mathbb{F}_q$ with basis

$$X = \{x_0, x_i, x_i' \colon 1 \leqslant i \leqslant 3\}.$$

Identify $\mathrm{GL}(V)$ with $\mathrm{GL}_7(q)$ with respect to X and set

$$U_4 = \langle x_1, x_2, x_1', x_2' \rangle, \qquad V_3 = \langle x_1, x_2, x_3 \rangle.$$

Let $\bar{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}_q$ and α the Frobenius automorphism of $\bar{\mathbb{F}}$, so that $\alpha(t) = t^p$ for all $t \in \bar{\mathbb{F}}$ and $\mathbb{F}_q$ is the fixed-point set $C_{\mathbb{F}_{q^2}}(\alpha^e)$ of α^e. View α as the Frobenius automorphism of $\mathbb{F}_q$, so that $|\alpha| = e$ in $\operatorname{Aut}(\mathbb{F}_q)$. Moreover, α induces a field automorphism and a semilinear map, denoted again by α, of $\mathrm{GL}(\bar{\mathbb{F}} \otimes V) = \mathrm{GL}_7(\bar{\mathbb{F}})$ and $\bar{\mathbb{F}} \otimes V$, respectively. Thus $\alpha(V_3) = V_3$ and $\alpha(U_4) = U_4$.

Let $K = \mathrm{SL}(V_3)$ be the subgroup of $\mathrm{GL}(V)$ defined by [5, (2.3)]. Following [5], let $g(t)$ and $r(1)$ be the elements of $\mathrm{GL}(V)$ whose matrices with respect to the basis $\{x_1, x_2', x_2, x_1'\}$ are

$$g(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{pmatrix}, \qquad r(1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and with respect to the basis $\{x_3, x_0, x'_3\}$ are

$$g(t) = \begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix}, \qquad r(1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then α fixes $\langle g(t)\rangle$ and $r(1)$ and so stabilizes

$$G = \big\langle K, g(t), r(1)\colon t \in \mathbb{F}_q \big\rangle \leqslant \mathrm{GL}(V),$$

which is $G_2(q)$ by [5, p. 205]. It follows from [18, Proposition 7.2] that α is a field automorphism of order e on G.

Identify G with its inner automorphism group $\mathrm{Inn}(G)$. Let

$$A = \mathrm{Aut}(G) = G : \langle \beta \rangle,$$

where β is a field automorphism of order e on G except when $p = 3$, in which case β is an extraordinary graph automorphism, whose square is the field automorphism. Suppose

$$\beta = \alpha \quad \text{or} \quad \beta^2 = \alpha$$

according as $p \neq 3$ or $p = 3$.

We may always suppose $B \in \mathrm{Blk}(G)$ has a non-cyclic defect group, so that $r \mid (q^2 - 1)$ (see [2, p. 25]). Let r^a or r^{a+1} be the exact power of r dividing $q^2 - 1$ according as r is odd or even, and let the sign ϵ be chosen so that $r^a \mid q - \epsilon$.

Suppose $\delta = +$ or $-$. Denote by $2_\delta^{1+2\gamma}$ the extraspecial 2-group of order $2^{1+2\gamma}$ and type δ. Given $n \in \mathbb{N}$, denote by D_{2n} a dihedral group of order $2n$, by E_{r^n} an elementary abelian group of order r^n, and by $\mathbb{Z}_n$ or simply n a cyclic group of order n.

Let $G_2(q^2) = C_{\overline{G}}(\alpha^{2e})$, $K(q^2) = \mathrm{SL}(\mathbb{F}_{q^2} \otimes V_3) = \mathrm{SL}_3(q^2) \leqslant G_2(q^2)$, $L_+ = K = \mathrm{SL}(V_3)$ and $L_- = C_{K(q^2)}(r(1)\alpha^e) \simeq \mathrm{SU}_3(q)$, where $\overline{G} = G_2(\overline{\mathbb{F}})$ and $r(1)$ acts on $K(q^2)$ by conjugation. Then $L_- \leqslant C_{G_2(q^2)}(r(1)\alpha^e) \simeq G_2(q)$ and in this case we identify G with $C_{G_2(q^2)}(r(1)\alpha^e)$. Let $K_\delta = N_G(L_\delta)$, so that $K_\delta = \langle L_\delta, r(1)\rangle$ and α stabilizes L_δ.

Let $\{a, b\}$ be a set of fundamental roots of the root system

$$\Sigma = \big\{\pm a, \pm(a+b), \pm(2a+b), \pm b, \pm(3a+b), \pm(3a+2b)\big\}$$

of type G_2 with a a short root. Then $K = \mathrm{SL}(V_3)$ is generated by three long root subgroups of G. We may suppose the root subgroup $X_{-a} = \{x_{-a}(t) = g(t)\colon t \in \mathbb{F}_q\}$ and $X_b = \{x_b(t)\colon t \in \mathbb{F}_q\}$, where $x_b(t) \in K$ such that

$$x_b(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

with respect to the basis $\{x_1, x_2, x_3\}$ of V_3. If $p = 3$, then $\beta(x_a(t)) = x_b(t^3)$, $\beta(x_b(t)) = x_a(t)$ for all $t \in \overline{\mathbb{F}}$, and β induces an action, denoted again by β, on the system Σ (see [24, p. 440]) such that $\beta(\pm a) = \pm b$, $\beta(\pm(a+b)) = \pm(3a+b)$ and $\beta(\pm(2a+b)) = \pm(3a+2b)$.

Let T be the diagonal subgroup of $\overline{G}$ with respect to X, so that T is a maximal torus of $\overline{G}$ (cf. [5, p. 254]) stabilized by α. Then

$$N_{\overline{G}}(T) = \langle T, \rho, \sigma, \tau \rangle,$$

where $\rho = r(1)\tau\sigma$ and the actions of σ and τ on the basis X are given as [5, p. 254],

$$\sigma = (x_1, x_2, x_3)(x_1', x_2', x_3'), \qquad \tau = (x_2, x_3, -x_2, -x_3)(x_2', x_3', -x_2', -x_3').$$

In particular, $w^\alpha = w$ for all $w \in \langle \rho, \sigma, \tau \rangle$. Thus $T = H_1 \times H_2$, where $H_i = \{h_i(t)\colon t \in \overline{\mathbb{F}}^{\#}\}$ such that as elements of $K = \mathrm{SL}_3(q)$,

$$h_1(t) = \operatorname{diag}\{t, t^{-1}, 1\}, \qquad h_2(t) = \operatorname{diag}\{1, t, t^{-1}\}.$$

Set $h_3(t) = h_1(t^{-1})h_2(t^{-1})$. Then ρ inverts T,

$$\sigma = \big(h_1(t), h_2(t), h_3(t)\big) \quad \text{and} \quad \tau = \big(h_1(t), h_3\big(t^{-1}\big)\big)\big(h_2(t), h_2\big(t^{-1}\big)\big).$$

Suppose $p = 3$ and $\omega_a = r(1)$. Then $\omega_a x_{-a}(t)\omega_a^{-1} = x_a(-t)$ and with respect to the bases $\{x_1, x_2', x_2, x_1'\}$ and $\{x_3, x_0, x_3'\}$

$$x_a(t) = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad x_a(t) = \begin{pmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ t^2 & t & 1 \end{pmatrix}.$$

So $\omega_a = x_a(1)x_{-a}(-1)x_a(1)$ and $\omega_b = \omega_a^\beta = x_b(1)x_{-b}(-1)x_b(1) = \tau$. Similarly, using $\omega_a x_b(t)\omega_a^{-1} = x_{3a+b}(t)$, $\omega_b x_{3a+b}(t)\omega_b^{-1} = x_{3a+2b}(t)$, $\omega_b x_a(t)\omega_b^{-1} = x_{a+b}(t)$, and $\omega_a x_{a+b}(t)\omega_a^{-1} = x_{2a+b}(t)$, we get the matrix representative of each root element with respect to the basis X.

As an element of K, $\sigma = \tau\tau_1$, where $\tau_1 = (x_1, x_2, -x_1, -x_2)(x_1', x_2', -x_1', -x_2')$. Thus $\tau_1 = x_{-3a-2b}(-1)x_{3a+2b}(1)x_{-3a-2b}(-1)$, $(\tau_1)^\beta = x_{-2a-b}(-1)x_{2a+b}(1)x_{-2a-b}(-1)$ and so

$$(\tau_1)^\beta = (x_2', x_3, -x_2', -x_3)(x_2, x_3', -x_2, -x_3')(x_1', x_1)(x_0, -x_0).$$

Since $\tau^\beta = r(1)$ and $\rho^\beta = r(1)^\beta\tau^\beta\sigma^\beta = \tau r(1)\sigma^\beta$, it follows that

$$\sigma^\beta = h_2(-1)\sigma^{-1}, \qquad \rho^\beta = h_3(-1)\rho, \qquad \tau^\beta = \rho\tau\sigma \tag{3.1}$$

and so $(\sigma\tau)^\beta = h_2(-1)\rho\sigma\tau$. In addition, since $h_a(t) = x_{-a}(t^{-1} - 1)x_a(1)x_{-a}(t - 1)x_a(-t^{-1})$ (cf. [24, p. 447]), it follows that

$$h_a(t) = h_1(t)h_2\big(t^2\big) = \operatorname{diag}\{t, t, t^{-2}\} \in K.$$

Similarly, $h_b(t) = h_2(t) \in K$. So $h_a(t)^\beta = h_b(t^3)$ and $h_b(t)^\beta = h_a(t)$ for all $t \in \overline{\mathbb{F}}$.

Let

$$Y = \left\{ \begin{pmatrix} P & \\ & P^{-t} \end{pmatrix} :\ P \in \mathrm{GL}(\mathbb{F}_{q^2}x_1 \oplus \mathbb{F}_{q^2}x_2) \right\} \leqslant \mathrm{SO}(\mathbb{F}_{q^2} \otimes U_4),$$

$J_+ = C_Y(\alpha^e) \simeq \mathrm{GL}_2(q)$, $J_- = C_Y(v\alpha^e) \simeq \mathrm{U}_2(q)$ and $J_\delta{}^* = h^{-1}J_\delta h \leqslant \mathrm{SO}(U_4)$, where $v \in C_{\mathrm{SO}(U_4)}(\alpha) = \mathrm{SO}_4^+(p)$ induces the inverse-transpose automorphism on Y and $h \in \mathrm{O}(U_4)$ is the reflection on $x_1 + x_1'$. Then $N_G(J_\delta{}^*) = \langle J_\delta{}^*, \rho\rangle$, $N_G(J_\delta) = \langle J_\delta, \rho\rangle$ and J_δ is not $G_2(q)$-conjugate to $J_\delta{}^*$.

Suppose L is a maximal torus of G. As shown in the proof of [5, (15.1)], L is conjugate in $\overline{G}$ to some $C_T(\alpha^e w)$, where $w \in \{1, \sigma, \tau, \rho, \rho\sigma, \rho\tau\}$. If $(w_+, w_2^+, w_3^+) = (1, \sigma, \tau)$ and $(w_-, w_2^-, w_3^-) = (\rho, \rho\sigma, \rho\tau)$, then $C_T(\alpha^e w_\delta) \simeq \mathbb{Z}_{q-\delta} \times \mathbb{Z}_{q-\delta}$, $C_T(\alpha^e w_2^\delta) \simeq \mathbb{Z}_{q^2+\delta q+1}$ and $C_T(\alpha^e w_3^\delta) \simeq \mathbb{Z}_{q^2-1}$.

Fix maximal tori T_δ and T_2^δ of L_δ such that T_δ is diagonal and isomorphic to $\mathbb{Z}_{q-\delta} \times \mathbb{Z}_{q-\delta}$, $T_2^\delta \simeq \mathbb{Z}_{q^2+\delta q+1}$, α stabilizes T_δ and T_2^δ,

$$N_G(T_\delta) = N_{K_\delta}(T_\delta) = \langle T_\delta, \sigma, \tau, \rho\rangle,$$

and $N_G(T_2^\delta) = N_{K_\delta}(T_2^\delta) = \langle T_2^\delta, \sigma, \rho\rangle$, where $\sigma, \tau, \rho \in C_G(\alpha) = G_2(p)$ are given above. Let T_3^+ and T_3^- be maximal tori of J_δ and $J_\delta{}^*$, respectively such that $T_3^\delta \simeq \mathbb{Z}_{q^2-1}$, α stabilizes T_3^δ and $N_G(T_3^\delta) = \langle T_3^\delta, \tau, \rho\rangle$.

Suppose $p = 3$. Then β stabilizes each T_δ and T_2^δ, and interchanges T_3^+ and T_3^-. In addition,

$$N_A(T_\delta) = \langle T_\delta, \sigma, \tau, \rho\rangle : \langle\beta\rangle \tag{3.2}$$

and moreover, $|\rho| = 2$, $|\sigma| = 3$, $\tau^2 = h_2(-1)$ and

$$\tau^{-1}\rho\tau = \rho, \qquad \sigma^{-1}\rho\sigma = h_1(-1)\rho, \qquad \tau^{-1}\sigma\tau = h_2(-1)\sigma^{-1}.$$

Note that the actions of τ, σ on T_δ are different from that given before [3, (2.5)].

Let $\iota : \overline{\mathbb{F}}^\# \to (\overline{\mathbb{Q}}_\ell^\#)_{p'}$ be the fixed group isomorphism in the Deligne–Lusztig theory. Then ι gives rise to a group isomorphism (cf. [8, Proposition 4.4.1]):

$$\phi : L \longrightarrow \mathrm{Irr}(L), \tag{3.3}$$

where $L \in \{T_\delta, T_2^\delta, T_3^\delta\}$. Following [8], we suppose β acts on $\mathrm{Irr}(L)$ by ${}^\beta(\phi(s))(t) = \phi(s)(t^\beta)$.

(3A) *In the notation above, suppose $s \in L$ and $R_L^G(\phi(s))$ is the Deligne–Lusztig generalized character. Then $R_L^G(\phi(s))^\beta = R_{\beta L\beta^{-1}}^G(\phi(s^\beta))$.*

Proof. The proof is similar to that of [16, (4A)] (see [23, pp. 181 and 257]). Let $\Gamma = \mathrm{Hom}(T, \overline{\mathbb{F}}^\#)$ and $\Gamma^* = \mathrm{Hom}(\overline{\mathbb{F}}^\#, T)$ be the character and cocharacter groups of T. There is a group isomorphism $\kappa : \Gamma \to \Gamma^*$ carries the roots to coroots. Thus $\iota^{-1}\phi(s) \in \mathrm{Hom}(L, \overline{\mathbb{F}}^\#)$ and hence it is the restriction of some $\lambda_1 \in \Gamma$. If $|\phi(s)| = n$, then for some $\lambda \in \Gamma$

$$\lambda\big(\alpha^e(t)w^{-1}t^{-1}w\big) = \lambda_1^n(t) \quad \text{for } t \in T,$$

If $\nu = \kappa(\lambda) \in \Gamma^*$, then the $\overline{G}$-class of $\nu(\iota^{-1}(\frac{1}{n}))$ is fixed by α^e, and its intersection with G is the G-class of s.

Since β is a Frobenius map (cf. [8, p. 31]), it follows that $\kappa({}^\beta\chi) = \kappa(\chi)^\beta$ for all $\chi \in \Gamma$ (cf. [8, Proposition 4.3.1]). The procedure associating $\phi(s)$, λ_1, λ and n to $R_L^G(\phi(s))$ associates ${}^\beta\phi(s)$, ${}^\beta\lambda_1$, ${}^\beta\lambda$ and n to $R_L^G(\phi(s))^\beta$. Since ${}^\beta\lambda$ corresponds to ν^β under κ, the $\overline{G}$-class of $\nu(\iota^{-1}(\frac{1}{n}))^\beta$ contains s^β and (3A) follows. □

Since $h_a = a^\vee$ and $h_b = b^\vee$ are coroots corresponding to roots a and b, respectively, it follows that $\kappa(a) = h_a$ and $\kappa(b) = h_b$. Moreover, if $p = 3$, then $({}^\beta a, {}^\beta b) = (b, 3a)$ and $(h_a^\beta, h_b^\beta) = (3h_b, h_a)$. By [8, Proposition 4.2.3], κ induces a group automorphism κ of $D_{12} = N_{\overline{G}}(T)/T$ such that $\kappa(w_{a_1}) = w_{\kappa(a_1)}$ for each root $a_1 \in \Sigma$, where w_{a_1} is the reflection with respect to a_1. In addition, $\kappa({}^w\chi) = \kappa(\chi)^{\kappa(w^{-1})}$ for all $\chi \in \Gamma$ and $w \in D_{12}$. Regard h_a and h_b as simple roots in the dual root system. Then h_a is long and h_b is short, so that $\kappa(w_a) = w_b$ and $\kappa(w_b) = w_a$.

Identify wT with w for $w \in N_{\overline{G}}(T)$, so $D_{12} = \langle \rho, \sigma, \tau \rangle$. By [24, Section 2], we may suppose $w_a = \rho\sigma^{-1}\tau$ and $w_b = \tau$. Thus $\sigma = (w_a w_b)^2$ and $\kappa(\sigma) = (w_b w_a)^2 = \sigma^{-1}$, so that $\kappa({}^\sigma\chi) = \kappa(\chi)^\sigma$. Similarly, $\kappa({}^\tau\chi) = \kappa(\chi)^{\rho\tau\sigma}$ and $\kappa({}^\rho\chi) = \kappa(\chi)^\rho$, since $\kappa(\tau) = \rho\tau\sigma$ and $\kappa(\rho) = \rho$. Apply a proof similar to that of [16, (4A)] to the group T. Thus

$$ {}^\sigma\phi(s) = \phi(s^\sigma), \quad {}^\tau\phi(s) = \phi(s^{\rho\tau\sigma}), \quad {}^\rho\phi(s) = \phi(s^\rho), \quad {}^\beta\phi(s) = \phi(s^\beta) \tag{3.4} $$

for all $s \in T_\delta$.

Suppose $p = 3$. Then $\beta\sigma^2$ interchanges J_δ and ${J_\delta}^*$, $(\beta\sigma^2)^2 = h_1(-1)\alpha$

$$ N_A(J_\delta) = N_G(J_\delta) : \langle\alpha\rangle \quad \text{and} \quad N_A({J_\delta}^*) = N_G({J_\delta}^*) : \langle\alpha\rangle. $$

Note that $Z(J_\delta) = \{\mathrm{diag}\{s, s, s^{-2}\} \in T_\delta \colon s \in \mathbb{Z}_{q-\delta}\} \leqslant L_\delta$ and $Z({J_\delta}^*) = \{\mathrm{diag}\{s, s^{-1}, 1\} \in T_\delta \colon s \in \mathbb{Z}_{q-\delta}\} \leqslant L_\delta$.

Since $J_\delta {J_\delta}^* = \mathrm{SO}(U_4)$ (cf. [5, (2.7)]), it follows that $\beta\sigma^2$ stabilizes $\mathrm{SO}(U_4)$ when $p = 3$. Thus

$$ N_A\big(\mathrm{SO}(U_4)\big) = \big\langle \mathrm{SO}(U_4), \beta' \big\rangle, $$

where $\beta' = \beta\sigma^{-1}$ or α according as $p = 3$ or $p \neq 3$. Moreover, if $p = 3$, then

$$ \sigma^{\beta'} = h_1(-1)\sigma^{-1}, \qquad \rho^{\beta'} = h_1(-1)\rho, \qquad \tau^{\beta'} = h_3(-1)\rho\sigma\tau $$

and so $(\tau\sigma)^{\beta'} = h_3(-1)\rho\tau\sigma$.

Suppose q is odd and $\Omega(U_4)$ is the only subgroup of $\mathrm{SO}(U_4)$ of index two. Then $\Omega(U_4)$ is the central product $M_1 \circ M_2$ of $M_1 = \mathrm{Sp}_2(q)$ and $M_2 = \mathrm{Sp}_2(q)$ over $Z(M_1) = Z(M_2)$. Let W_i be the underlying (symplectic) space of M_i. Then $M_i = \mathrm{SL}(W_i)$ and we may suppose $\mathrm{SO}(U_4) = \mathrm{SO}(W_1 \otimes W_2)$. Let $g_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{GL}(W_1)$ and $g_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{GL}(W_2)$. Then $g = g_1 \otimes g_2 \in \mathrm{SO}(U_4) \backslash \Omega(U_4)$ and

$$ \mathrm{SO}(U_4) = \langle M_1 \circ M_2, g \rangle. \tag{3.5} $$

(3B) *Let $M = \mathrm{SO}(U_4) = \mathrm{SO}_4^+(q)$ and $\Omega = \Omega(U_4) = \Omega_4^+(q)$. Suppose p is odd and $\Omega = M_1 \circ M_2$, where $M_1 \simeq M_2 \simeq \mathrm{SL}_2(q)$.*

(a) *If* $e \geqslant 2$, *then* α *induces a field automorphism* α_{M_i} *of order* e *on* M_i *for each* i.
(b) If $p = 3$, *then* β' *interchanges* M_1 *and* M_2.

Proof. Let $s_i \in M_i$ be an element of order 2^a, so that $C_{M_i}(s_i) = \mathbb{Z}_{q-\epsilon}$, $J_i = C(s_i) \simeq \mathrm{GL}_2^\epsilon(q)$, $J_1 \geqslant C_\Omega(s_1) = C_{M_1}(s_1) \circ M_2$ and $J_2 \geqslant C_\Omega(s_2) = M_1 \circ C_{M_2}(s_2)$. Then J_1 and J_2 are not G-conjugate. Indeed, if $J_1^w = J_2$ for some $w \in G$, then $\Omega_1(\langle s_1\rangle)^w = \Omega_1(\langle s_2\rangle)$, $w \in M = C_G(Z(M))$, $w = t$ or tg for some $t \in M_1 \circ M_2$, and so $s_1^w \in M_1$, $s_1^w \in M_1 \cap M_2 = Z(M)$, which is impossible.

Since G: $\langle\alpha\rangle$ has exactly two classes of subgroups J_ϵ, it follows that $J_i^{c_i\alpha} = J_i$ for some $c_i \in G$, $\Omega_1(\langle s_i\rangle)^{c_i\alpha} = \Omega_1(\langle s_i\rangle)$ and $c_i \in M$, since α centralizes $Z(M) = \Omega_1(\langle s_i\rangle)$. But M_i is the commutator group $[J_i, J_i]$, $c_i = t$ or tg for some $t \in \Omega$, so $M_i^{c_i\alpha} = M_i^\alpha = M_i$. If $p = 3$, then A has only one class of subgroups J_ϵ and so J_1 is A-conjugate to J_2. Thus $s_1^\omega = s_2$ for some $\omega \in A\backslash G$, and so $Z(M)^\omega = Z(M)$, $J_1^\omega = J_2$ and $\omega \in \langle M, \beta'\rangle\backslash M{:}\,\langle\alpha\rangle$. Since Ω is unique in M, it follows that $C_\Omega(s_1)^\omega = C_\Omega(s_2)$. Let $Z_j = [C_\Omega(s_j), C_\Omega(s_j)]$ for $j = 1, 2$. Then $Z_1^\omega = Z_2$ and $Z_j \leqslant M_i$, where $i \neq j$ and $i, j \in \{1, 2\}$. If $q \neq 3$, then $Z_j = M_i$. Suppose $q = 3$, so that $Z_j = O_2(M_i)$ is quaternion. If m_2 is an element of order 3 in $\mathrm{SL}(W_2) \leqslant C_\Omega(s_1)$, then $m_2^\omega \in C_\Omega(s_2) = \mathrm{SL}(W_1) \circ \langle s_2\rangle$, so that $m_2^\omega \in \mathrm{SL}(W_1)$. But $\langle Z_1, m_2\rangle = \mathrm{SL}(W_2)$, so $\mathrm{SL}(W_2)^\omega = \mathrm{SL}(W_1)$. In all cases, $M_2^\omega = M_1$. Now $\omega = t\beta'$ for some $t \in M{:}\,\langle\alpha\rangle$, so $M_1 = M_2^\omega = M_2^{\beta'}$ and $M_1^{\beta'} = M_2$. This proves (b).

Suppose $e \geqslant 2$ and e' is a proper factor of e. Then $C_M(\alpha^{e'}) = \mathrm{SO}_4^+(p^{e'})$, $C_\Omega(\alpha^{e'}) = \Omega_4^+(p^{e'})$ and $C_{M_i}(\alpha^{e'}) \neq M_i$. Thus the induced automorphism of $\alpha^{e'}$ on M_i is non-trivial and so $|\alpha_{M_i}| = e$. Since $\mathrm{Aut}(M_i)$ consists of only field, diagonal and inner automorphisms of M_i, it follows from [18, Proposition 7.2] that α_{M_i} is a field automorphism of M_i. $\square$

Let $\mathbb{Z}_{r^a} = Z(J_\epsilon)$ and $\mathbb{Z}^*_{r^a} = Z(J_\epsilon{}^*)$. Following [3], we define radical r-chains $C(1)$ and $C(2)$ as follows:

$$C(1){:}\ \begin{cases} 1 < \mathbb{Z}_{r^a} < O_r(T_\epsilon) & \text{if } r \geqslant 5,\\ 1 < Z(L_\epsilon) & \text{if } r = 3,\end{cases} \qquad C(2){:}\ 1 < \mathbb{Z}^*_{r^a} < O_r(T_\epsilon) \quad \text{if } r \geqslant 3.$$

If $r \geqslant 5$, then we may take $\mathcal{R}(G)/G = \{C(1)_0, C(1)_1, C(2)_1, C(1)^1, C(1), C(2)\}$ and set $\mathcal{R}^0(G) = \mathcal{R}(G)$. If $r = 3$, then let $\mathcal{R}^0(G)$ be a G-invariant subfamily of $\mathcal{R}(G)$ such that

$$\mathcal{R}^0(G)/G = \bigl\{C(1)_0, C(1), C(2)_1, C(2)\bigr\}.$$

(3C) *If* $r \geqslant 3$ *and* $B \in \mathrm{Blk}(G)$ *with* $\mathrm{d}(B) > 0$, *then for* $d, u, w \in \mathbb{Z}$

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|}\mathrm{k}\bigl(N(C), B, d, [w]\bigr) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|}\mathrm{k}\bigl(N(C), B, d, [w]\bigr).$$

Proof. Suppose $r = 3$, so that $p \neq 3$. If R is a nonabelian radical subgroup of G or $R = \mathbb{Z}_{3^a}$, then we may suppose $Z(L_\epsilon) = \Omega_1(Z(R))$ and so $N_A(R) \leqslant N_A(Z(L_\epsilon))$. If $R = O_3(T_\epsilon)$, then $N_A(R) = N_G(T_\epsilon){:}\,\langle\alpha\rangle \leqslant N_A(Z(L_\epsilon))$. Thus we can apply the proof of [3, (2A)] with some obvious modifications. $\square$

Suppose $r = 2$, so that p is odd. Thus $G_2(p) = C_G(\alpha)$ has a unique class of subgroups E_8 such that $C_G(E_8) = E_8$ and $N_G(E_8) = N_{G_2(p)}(E_8) = E_8.L_3(2)$. We may suppose $E_8 =$

$\langle \rho, \Omega_1(O_2(T_\epsilon))\rangle$. If $p = 3$, then β induces a non-trivial automorphism of $G_2(3)$, and by [10, p. 61], $N_A(G_2(3)) = G_2(3){:}\,2 \times \langle\alpha\rangle$ and $N_A(E_8) = E_8.L_3(2){:}\,2 \times \langle\alpha\rangle$.

Let $\Phi(G, r)$ be a set of representatives for conjugacy G-classes of radical r-subgroups. If $q \neq 3$, then $\Phi(G, 2)$ is given in [3, Section 2]. If $q = 3$, then the set $\Phi(G, 2)$ given in [3, Section 2] is incorrect (see the remark after (3F)). In this case we may take (see [2, Section 2])

$$\Phi(G,2) = \bigl\{1, E_8, \bigl\langle O_2(T_\epsilon), \rho\bigr\rangle, 2_+^{1+4}, S\bigr\},$$

where $2_+^{1+4} = O_2(\mathrm{SO}_4^+(4))$ and $S \in \mathrm{Syl}_2(G)$. Following [3], we define 2-chains $C(i)$ for $1 \leqslant i \leqslant 4$ as follows.

$$C(1){:}\ 1 < \mathbb{Z}_2 < O_2(T_{-\epsilon}),$$
$$C(2){:}\ \begin{cases} 1 < \mathbb{Z}_2 < O_2(T_\epsilon) & \text{if } 2^a \neq q - \epsilon, \\ 1 < \langle O_2(T_\epsilon), \rho\rangle < S' & \text{if } 2^a = q - \epsilon, \end{cases}$$
$$C(3){:}\ 1 < E_8 < 2_+^{1+4}, \quad C(4){:}\ 1 < E_8 < W < S'',$$

where $\mathbb{Z}_2 = Z(\mathrm{SO}(U_4))$, $2_+^{1+4} \leqslant N_G(E_8)$, $E_8 = \langle \rho, \Omega_1(O_2(T_\epsilon))\rangle$, $W = \langle \rho, \Omega_2(O_2(T_\epsilon))\rangle$, $S'' = \langle \Omega_2(O_2(T_\epsilon)), \rho, \tau\sigma\rangle \in \mathrm{Syl}_2(N_G(E_8))$ and $S' = \langle O_2(T_\epsilon), \rho, \tau\sigma\rangle \in \mathrm{Syl}_2(N_G(T_\epsilon))$. Note that if $q = 3$, then $S' = S'' \in \mathrm{Syl}_2(G)$. Let $\mathcal{R}^0(G)$ be a G-invariant subfamily of $\mathcal{R}(G)$ such that if $q \neq 3$, then

$$\mathcal{R}^0(G)/G = \bigl\{C(1)_0, C(1)_1, C(1)^1, C(1), C^1(2), C(2), C(3)_1, C(3), C(4)_2, C(4)\bigr\},$$

where $C^1(2) = C(2)^1$ or $C(2)_1$ according as $2^a \neq q - \epsilon$ or $2^a = q - \epsilon$; if $q = 3$, then

$$\mathcal{R}^0(G)/G = \bigl\{C(1)_0, C(3)_1, C(3)^1, C(3)\bigr\}. \tag{3.6}$$

We have the following proposition.

(3D) *Suppose B is a 2-block of $G = G_2(q)$ with $\mathrm{d}(B) > 0$ and $d, u \in \mathbb{Z}$. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|}\mathrm{k}\bigl(N_G(C), B, d, u\bigr) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|}\mathrm{k}\bigl(N_G(C), B, d, u\bigr).$$

Proof. Suppose $C \in \mathcal{R}(G)$ is given by (2.1) with $|C| \geqslant 1$ and $P_1 \in \Phi(G, 2)$. We sketch the proof (1) below because it is a modification of that of [3, (2B)].

(1) Suppose $P_1 = \mathbb{Z}_2$ and let $\Omega^* = \{1, \mathbb{Z}_2, O_2(T_\epsilon), E_8, O_2(T_{-\epsilon}), \langle O_2(T_\epsilon), \rho\rangle\}$. If $R \in \Phi(G, 2)\backslash\Omega^*$, then $\mathbb{Z}_2 = \Omega_1(Z(R))$ and $N_A(R) \leqslant N_A(\mathbb{Z}_2) = \langle \mathrm{SO}(U_4), \beta'\rangle$. The proof (1) of [3, (2B)] can be apply here with some obvious modifications. So we may take $P_1 \in \Omega^*$ and if $P_1 = \mathbb{Z}_2$ with $|C| \geqslant 2$, then $q \neq 3$ and we may suppose $P_2 \in \{O_2(T_{-\epsilon}), O_2(T_\epsilon)\}$.

If $P_1 = O_2(T_\epsilon)$, then

$$N_A\bigl(\bigl\langle O_2(T_\epsilon), \rho\bigr\rangle\bigr) = \bigl\langle O_2(T_\epsilon), \rho, \sigma, \tau, \beta\bigr\rangle \leqslant N_A(P_1) = N_A(T_\epsilon).$$

Apply the proof (2) of [3, (2B)] with some obvious modifications. Then we may suppose P_1 and $\langle O_2(T_\epsilon), \rho\rangle$ are not G-conjugate, and if $P_1 = O_2(T_\epsilon)$ with $|C| \geqslant 2$, then $P_2 \in \{\mathbb{Z}_{2^a} \wr \mathbb{Z}_2, (\mathbb{Z}_{2^a} \wr$

$\mathbb{Z}_2)^*, S'\}$, where $\mathbb{Z}_{2^a} \wr \mathbb{Z}_2 = \langle O_2(T_\epsilon), \tau\rangle$, $(\mathbb{Z}_{2^a} \wr \mathbb{Z}_2)^* = \langle O_2(T_\epsilon), \rho\tau\sigma\rangle$, and $S' = \langle O_2(T_\epsilon), \rho, \tau\sigma\rangle$. But $N_A(T_\epsilon) \cap N_A(P_2) \leqslant \langle T_\epsilon, \rho, \tau\sigma, \beta'\rangle \leqslant N_A(\mathbb{Z}_2)$, so the proof (3) of [3, (2B)] can be applied here with some obvious modifications. It follows that the remaining chains $C \in \mathcal{R}(G)$ such that P_1 is conjugate to $\mathbb{Z}_2$ or $O_2(T_\epsilon)$ and $|P_2| \neq 4$ have representatives, $C(1)_1$, $C(2)$ and $C(2)^1$.

(2) Suppose $P_1 = E_8$, so that $N_G(E_8) = N_{C_G(\alpha)}(E_8)$. If $p \neq 3$, then $N_A(E_8) = N(E_8) \times \langle\alpha\rangle$. It follows by the proof (4) of [3, (2B)] that remaining chains in $\mathcal{R}(G)$ with $P_1 \simeq E_8$ have representatives $C(3)$, $C(3)_1$, $C(4)$ and $C(4)_2$.

Suppose $p = 3$. As shown in the proof (4) of [3, (2B)] we may take $P_2 \in \{2_+^{1+4}, W, S''\}$. So $N_{N(E_8)}(W) = \langle \Omega_2(O_2(T_\epsilon)), \sigma, \tau, \rho\rangle$, $N_{N_A(E_8)}(W) = N_{N(E_8)}(W){:}\langle\beta\rangle$, and $N_{N_A(E_8)}(S'') = \langle S'', \beta'\rangle$.

Let $M = C_{\mathrm{SO}(U_4)}(\alpha)$. In the notation of (3.5) (with $q = 3$), let

$$v_i = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad y_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad m_i = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

be elements of $M_i = \mathrm{SL}_2(3)$. Then we may suppose

$$E_8 = \langle z, v_1v_2, y_1y_2\rangle \leqslant O_2(M) = 2_+^{1+4} \leqslant N(E_8).$$

Now both $m = m_1 \otimes m_2$ and g normalize E_8, and $|\mathrm{SO}_4^+(3){:}\langle O_2(M), m, g\rangle| = 3$, so $N_{\mathrm{SO}_4^+(3)}(E_8) = \langle O_2(M), m, g\rangle$. Since $N(E_8) \leqslant C_G(\alpha)$, it follows that

$$N_{N(E_8)}\bigl(2_+^{1+4}\bigr) = N(E_8) \cap N\bigl(2_+^{1+4}\bigr) = N_{\mathrm{SO}_4^+(3)}(E_8) = \bigl\langle 2_+^{1+4}, m, g\bigr\rangle, \tag{3.7}$$

and $N_{N_A(E_8)}(2_+^{1+4}) = \langle N_{N(E_8)}(2_+^{1+4}), \beta'\rangle$. We may suppose

$$\Phi\bigl(N(E_8), 2\bigr) = \bigl\{E_8, O_2\bigl(C_{\mathrm{SO}(U_4)}(\alpha)\bigr), \bigl\langle \Omega_2\bigl(O_2(T_\epsilon)\bigr), \rho\bigr\rangle, S''\bigr\}.$$

Let C': $1 < E_8 < 2_+^{1+4} < S''$ and $g(C')$: $1 < E_8 < S''$. Then $N(C') = N(g(C')) = S''$ and $N_A(C') = N_A(g(C')) = \langle S'', \beta'\rangle$. Thus

$$\mathrm{k}\bigl(N(C'), B, d, u\bigr) = \mathrm{k}\bigl(N\bigl(g(C')\bigr), B, d, u\bigr). \tag{3.8}$$

The remaining chains in $\mathcal{R}(G)$ with $P_1 \simeq E_8$ have representatives $C(3)$, $C(3)_1$, $C(4)$ and $C(4)_2$.

If $q = p = 3$, then we can suppose $S'' = S' = S \in \mathrm{Syl}_2(G)$. If $C' = C(4)$ and $g(C') = C(2)$, then $N(C') = N(g(C')) = S$ and $N_A(C') = N_A(g(C')) = \langle S, \beta'\rangle$, so that (3.8) still holds. Similarly, we can apply (3.8) to $C' = C(2)_1$ and $g(C') = C(4)_2$, since $\langle O_2(T_\epsilon), \rho\rangle = W = \langle \Omega_2(T_\epsilon), \rho\rangle$.

(3) If $P_1 = O_2(T_{-\epsilon})$, then $q \neq 3$ and $N(P_1) = N(T_{-\epsilon}) = \langle T_{-\epsilon}, \sigma, \tau, \rho\rangle$. Let $Q = \langle E_8, \tau\sigma\rangle \in \mathrm{Syl}_2(N(P_1))$, C': $1 < O_2(T_{-\epsilon}) < E_8$ and $g(C')$: $1 < O_2(T_{-\epsilon}) < E_8 < Q$. By [15, Corollary V.3.11], $\mathrm{Irr}(b(C')) = \mathrm{Irr}(N(C'))$ and $\mathrm{Irr}(b(g(C'))) = \mathrm{Irr}(N(g(C')))$, where $b(C') = B_0(N(C'))$ and $b(g(C')) = B_0(N(g(C')))$ are the principal blocks. Let $h_i = h_i(-1)$ for $i = 1, 2, 3$, so that $O_2(T_{-\epsilon}) = \langle h_1, h_2\rangle$. Since $\rho^\sigma = h_1\rho = \rho^{\tau\sigma}$ and $h_2^\sigma = h_2^{\tau\sigma} = h_3$, it follows that $N(g(C')) = Q = \mathbb{Z}_2 \times D_8$, $N_A(g(C')) = \langle Q, \beta'\rangle$, $N(C') = \langle Q, \sigma\rangle = \mathbb{Z}_2 \times A_4{:}2$ and $N_A(C') = \langle Q, \sigma, \beta'\rangle$, where $\mathbb{Z}_2 = \langle h_2\rho\rangle$, $D_8 = \langle h_1, h_2, \tau\sigma\rangle$ and $A_4{:}2 = \langle h_1, h_2, \sigma, \tau\sigma\rangle$. Thus $\mathrm{Irr}(Q)$ has 8 linear characters and 2 characters of degree 2, and $\mathrm{Irr}(\langle Q, \sigma\rangle)$ has 4 linear characters, 2 characters of degree 2 and

4 of degree 3. If $p \neq 3$, then $N_A(g(C')) = Q \times \langle\alpha\rangle$ and $N_A(C') = \langle Q, \sigma\rangle \times \langle\alpha\rangle$, so that (3.8) holds.

Suppose $p = 3$, so that $\beta'^2 = h_1\alpha$ and α acts trivially on $\langle Q, \sigma\rangle$. Let $\theta = \beta'\langle\alpha\rangle \in \langle\beta'\rangle/\langle\alpha\rangle$. Then $\theta^2 = h_1$ and

$$\rho^\theta = h_1\rho, \qquad h_2^\theta = h_3, \qquad (\tau\sigma)^\theta = h_3\rho\tau\sigma, \qquad \sigma^\theta = h_1\sigma^{-1}.$$

Let $H = \langle h_1, h_2, \rho, \tau, \sigma, \theta\rangle = \langle Q, \sigma, \theta\rangle$ and $h = \theta\tau\sigma h_2$. Then $h^2 = h_2\rho$ and $C_H(h) = \langle h, h_1, h_2, \sigma\rangle = \mathbb{Z}_4 \times A_4$, where $\mathbb{Z}_4 = \langle h\rangle$ and $A_4 = \langle h_1, h_2, \sigma\rangle$. Moreover, $h^\theta = \theta\tau\sigma\rho = h^{-1}$, $A_4^\theta = A_4$ and $H = \langle C_H(h), \theta\rangle$ with $(H : C_H(h)) = 2$. Since $[H, H] = \langle h_1, \sigma, h_2, \rho\rangle$ and $h^\theta = h^{-1}$, it follows that $\mathrm{Irr}(H)$ has 4 linear characters, 4 characters of degree 3, 1 of degree 6, 5 of degree 2, and θ stabilizes exactly two linear characters, two characters of degree 3 and 2 of degree 2 in $\mathrm{Irr}(N(C'))$.

If $N = \langle h_1, h_2, \tau\sigma, \rho, \theta\rangle$, then $C_N(h) = \langle h, h_1, h_2\rangle = \mathbb{Z}_4 \times O_2(T_{-\epsilon})$, $N = \langle C_N(h), \theta\rangle$ and $[N, N] = \langle h_1, h_3\rho\rangle$. Thus $\mathrm{Irr}(N)$ has exactly 8 linear characters, 6 of degree 2 and θ stabilizes exactly 4 linear characters and 2 characters of degree 2 in $\mathrm{Irr}(N(g(C')))$. It follows that

$$\mathrm{k}\big(N(C'), B_0, d, u\big) = \mathrm{k}\big(N\big(g(C')\big), B_0, d, u\big) = \begin{cases} 4 & \text{if } d = 4 \text{ and } u = 2e, \\ 4 & \text{if } d = 4 \text{ and } u = e, \\ 2 & \text{if } d = 3 \text{ and } u = 2e, \\ 0 & \text{otherwise.} \end{cases}$$

We may take $P_2 \in \{D_8, D_8^*, Q\}$, where $D_8 = \langle P_1, \tau\rangle$ and $D_8^* = \langle P_1, \rho\tau\sigma\rangle$. Let

$$C'\colon\ 1 < O_2(T_{-\epsilon}) < P_2 < \cdots < P_n \in \mathcal{R}(G),$$
$$g(C')\colon\ 1 < \mathbb{Z}_2 < O_2(T_{-\epsilon}) < P_2 < \cdots < P_n.$$

Then $N(C') = N(g(C'))$ and $N_A(C') = N_A(g(C'))$, since $N(T_{-\epsilon}) \cap N(P_2) \leqslant T_{-\epsilon}Q \leqslant N(\mathbb{Z}_2)$ and since $N_A(T_{-\epsilon}) \cap N_A(P_2) \leqslant \langle T_{-\epsilon}, Q, \beta'\rangle \leqslant N_A(\mathbb{Z}_2)$. So (3.8) holds. The remaining radical chains $C \in \mathcal{R}(G)$ such that P_1 is conjugate to $\mathbb{Z}_2$ or $O_2(T_{-\epsilon})$ and P_2 is not conjugate to $O_2(T_\epsilon)$ have representatives, $C(1)_1$, $C(1)$ and $C(1)^1$. $\square$

(3E) *Let $C \in \mathcal{R}^0(G)$ with $|C| \geqslant 1$. If $r \geqslant 3$, then*

C		$N_A(C)$	*Conditions*
$C(1)_1$	$1 < \mathbb{Z}_{r^a}$	$\langle \mathrm{GL}_2^\epsilon(q), \rho, \alpha\rangle$	$r \geqslant 5$
$C(1)^1$	$1 < O_r(T_\epsilon)$	$N_A(T_\epsilon)$	$r \geqslant 5$
$C(2)_1$	$1 < \mathbb{Z}_{r^a}^*$	$\langle \mathrm{GL}_2^\epsilon(q), \rho, \alpha\rangle$	$r \geqslant 3$
$C(1)$	$1 < \mathbb{Z}_{r^a} < O_r(T_\epsilon)$	$\langle T_\epsilon, \rho, \tau, \alpha\rangle$	$r \geqslant 5$
$C(2)$	$1 < \mathbb{Z}_{r^a}^* < O_r(T_\epsilon)$	$\langle T_\epsilon, \rho, \tau, \alpha\rangle$	$r \geqslant 3$
$C(1)$	$1 < Z(L_\epsilon)$	$\langle L_\epsilon, \rho, \alpha\rangle$	$r = 3$

where $\rho, \sigma, \tau, \alpha$ are given by (3.2).

Proof. It follows by the proof of (3C). $\square$

(3F) *Let $C \in \mathcal{R}^0(G)$ with $|C| \geqslant 1$. If $r = 2$, then*

C		$N_A(C)$	*Conditions*
$C(1)_1$	$1 < \mathbb{Z}_2$	$\langle \mathrm{SO}_4^+(q), \beta' \rangle$	$q \neq 3$
$C(1)^1$	$1 < O_2(T_{-\epsilon})$	$N_A(T_{-\epsilon})$	$q \neq 3$
$C(2)_1$	$1 < O_2(T_\epsilon)$	$N_A(T_\epsilon)$	$q - \epsilon \neq 2^a$
$C(2)_1$	$1 < \langle O_2(T_\epsilon), \rho \rangle$	$N_A(T_\epsilon)$	$q - \epsilon = 2^a$
$C(3)_1$	$1 < E_8$	$\langle E_8.L_3(2), \beta' \rangle$	none
$C(3)^1$	$1 < 2_+^{1+4}$	$\langle \mathrm{SO}_4^+(3), \beta' \rangle$	$q = 3$
$C(1)$	$1 < \mathbb{Z}_2 < O_2(T_{-\epsilon})$	$\langle T_{-\epsilon}, \rho, \tau\sigma, \beta' \rangle$	$q \neq 3$
$C(2)$	$1 < \mathbb{Z}_2 < O_2(T_\epsilon)$	$\langle T_\epsilon, \rho, \tau\sigma, \beta' \rangle$	$q - \epsilon \neq 2^a$
$C(2)$	$1 < \langle O_2(T_\epsilon), \rho \rangle < S'$	$\langle S', \beta' \rangle$	$q - \epsilon = 2^a$
$C(3)$	$1 < E_8 < 2_+^{1+4}$	$\langle 2_+^{1+4}.S_3, \beta' \rangle$	none
$C(4)_2$	$1 < E_8 < W$	$\langle W, \sigma, \tau, \beta \rangle$	$q \neq 3$
$C(4)$	$1 < E_8 < W < S''$	$\langle S'', \beta' \rangle$	$q \neq 3$

where ρ, σ, τ are given by (3.2), $\beta' = \alpha$ *or* $\beta\sigma^2$ *according as* $p \neq 3$ *or* $p = 3$.

Proof. It follows by the proof of (3D). □

Remark. If $q = 3$, the classification of radical 2-chains given by [3, (2.8)] is incorrect because some of the 2-chains are not radical. The family defined by [3, (2.8)] should be replaced by the family $\mathcal{R}^0(G)$ defined by (3.6). However, the proof of the main theorem [3, (4C)] still works if we replace the chain C: $1 < \mathbb{Z}_2$ by C: $1 < 2_+^{1+4}$, since $N(1 < \mathbb{Z}_2) = N(1 < 2_+^{1+4}) = \mathrm{SO}_4^+(3)$ (see also the proofs in Section 5).

4. The proof of the conjecture for odd primes

The notation and terminology of Sections 2 and 3 are continued in this section. We shall identify a dual group of $G = G_2(q)$ with G. Let $\mathcal{E}(G, (s))$ be the set of the irreducible constituents of Deligne–Lusztig generalized characters associated with the conjugacy class (s) of a semisimple element $s \in G$, and let

$$\mathcal{E}_r\big(G, (s)\big) = \bigcup_y \mathcal{E}\big(G, (sy)\big),$$

where $s \in G$ is a semisimple r'-element and y runs over all the r-elements of $C_G(s)$.

Let $\mathrm{Blk}^0(G, r)$ be the set of r-blocks of G with non-cyclic defect groups. By [19,20] and [21],

$$\mathrm{Blk}^0(G, r) = \begin{cases} \{B_1, B_2, B_3, B_a, B_b, B_{X_\gamma}\} & \text{if } r \geqslant 3, \\ \{B_1, B_3, B_{1a}, B_{1b}, B_{2a}, B_{2b}, B_{X_1}, B_{X_2}\} & \text{if } r = 2, \end{cases} \tag{4.1}$$

where $B_1 = B_0(G)$ is the principal block and $\gamma = 1$ or 2 according as $\epsilon = +$ or $-$. In addition, if $p = 2$, then there is no block B_2. If $p = 3$, then there is no block B_3.

Let $Q \leqslant G$, $B \in \mathrm{Blk}(G, r)$ and

$$\mathcal{Y}(B, Q) = \mathrm{Irr}(B) \cap \left(\bigcup_t \mathcal{E}\big(G, (t)\big) \right), \tag{4.2}$$

where t runs over G such that $C_G(t) = Q$. Denote by $\chi_{t,\mu}$ the character of $\mathcal{E}(G, (t))$ labelled by (t, μ), where μ is a unipotent irreducible character of $C_G(t)$. It follows by (3A) that $\chi_{t,\mu}^{\beta} \in \mathcal{E}(G, (t^{\beta}))$.

(4A) *Let $G = G_2(q)$ and $B \in \mathrm{Blk}^0(G, r)$.*

(a) *In the notation of* [19]*, if $p = 3$, then $(X_{13}^{\beta}, X_{23}^{\beta}) = (X_{14}, X_{24})$ and β stabilizes the other characters of $\mathcal{Y}(B_1, G)$ and $\mathcal{Y}(B_2, \mathrm{SO}(U_4))$.*
(b) *If $p \neq 3$ or $p = 3$ and $t^2 \neq 1$, then*

$$\chi_{t,\mu}^{\beta} = \chi_{t^{\beta},\mu}$$

for all $\chi_{t,\mu} \in (\mathcal{E}(G, (t)) \cap \mathrm{Irr}(B)) \backslash \{X_{19}, \overline{X}_{19}\}$.

Proof. Suppose $p = 3$, so that $r \neq 3$. If $t^2 = 1$, then $t = 1$ or $|t| = 2$, so that $\mathcal{E}(G, (t))^{\beta} = \mathcal{E}(G, (t))$. In the notation of [12], β fuses the classes A_2 and A_{31} containing $x_{3a+2b}(1)$ and $x_{2a+b}(1)$, respectively. In addition, $X_{13} = \theta_3$, $X_{14} = \theta_4$, $X_{23} = \theta_8$ and $X_{24} = \theta_9$ (see [19, Appendix B]). By [12, Table VII-2], $\theta_3^{\beta} = \theta_4$, $\theta_8^{\beta} = \theta_9$ and $\theta_i^{\beta} = \theta_i$ for $0 \leqslant i \leqslant 11$ with $i \notin \{3, 4, 8, 9\}$. But $\mathcal{Y}(B_1, G) \subseteq \{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_{10}, \theta_{11}\}$, so (a) holds.

Suppose $p = 2$. In the notation of [13], $\theta_4(A_2) = 0$ and $\theta_3^{\alpha}(A_2) = \theta_3(A_2) = q$. Thus $\theta_3^{\alpha} \neq \theta_4$. The rest of (b) can be proved similarly. $\square$

Let $M_\epsilon = N_G(T_\epsilon) = \langle T_\epsilon, \rho, \tau, \sigma \rangle$ and $N(\phi(t))$ the stabilizer of $\phi(t) \in \mathrm{Irr}(T_\epsilon)$ in M_ϵ, where ϕ is given by (3.3). A proof similar to that of [3, (4.2)] shows that we may suppose

$$N\big(\phi(t)\big) \in \big\{M_\epsilon, \langle T_\epsilon, \rho, \tau\sigma \rangle, \langle T_\epsilon, \tau, \sigma \rangle, \big\langle T_\epsilon, \rho^j \tau \sigma^i \big\rangle, j, i = 0, 1\big\},$$

since we consider only the characters in $\mathrm{Irr}(M_\epsilon)$. Let $\mathcal{S}(t)$ be the subset of $\mathrm{Irr}(N(\phi(t)))$ consisting of characters which cover $\phi(t)$.

(4B) *Let $G = G_2(q)$ and $t \in T_\epsilon$. If $p = 3$ and $\phi(t)^{\beta} = \phi(t)$, then $t^2 = 1$.*

(a) *Suppose $t = 1$. Then $\mathcal{S}(t) = \{\xi_{1,i}\colon 1 \leqslant i \leqslant 6\}$, where $\xi_{1,i}(1) = 1$ for $1 \leqslant i \leqslant 4$ and $\xi_{1,5}(1) = \xi_{1,6}(1) = 2$. If $p = 3$, then we may suppose $\xi_{1,3}^{\beta} = \xi_{1,4}$ and $\xi_{1,i}^{\beta} = \xi_{1,i}$ for all $i \neq 3, 4$. If $p \neq 3$, then α stabilizes each character of $\mathcal{S}(t)$.*
(b) *Suppose $|t| = 2$ and $N(\phi(t)) = \langle T_\epsilon, \rho, \tau\sigma \rangle$. Then $\mathcal{S}(t) = \{\xi_{t,i}\colon 1 \leqslant i \leqslant 4\}$, where $\xi_{t,i}(1) = 1$ for all i. If $p = 3$, then we may suppose $\xi_{t,3}^{\beta'} = \xi_{t,4}$ and $\xi_{t,i}^{\beta'} = \xi_{t,i}$ for all $i \neq 3, 4$. If $p \neq 3$, then α stabilizes each character of $\mathcal{S}(t)$.*
(c) *Suppose $|t| = 3$ and $C_G(t) \simeq L_\epsilon$. Then $p \neq 3$ and $\mathcal{S}(t) = \{\xi_{t,i}\colon 1 \leqslant i \leqslant 3\}$, where $\xi_{t,1}(1) = \xi_{t,2}(1) = 1$ and $\xi_{t,3}(1) = 2$. In addition, $\xi_{t,i}^{\alpha} = \xi_{t,i}$ for all i.*
(d) *Suppose $C_G(t) \simeq \mathrm{GL}_2^{\epsilon}(q)$. Then $\mathcal{S}(t) = \{\xi_{t,1}, \xi_{t,2}\}$, where both $\xi_{t,1}$ and $\xi_{t,2}$ are linear. In addition, $\xi_{t,i}^{\alpha} = \xi_{t^{\alpha},i}$ for $i = 1, 2$.*
(e) *Suppose $C_G(t) = T_\epsilon$. Then $\mathcal{S}(t) = \{\xi_{t,1}\}$, where $\xi_{t,1}$ is linear. In addition, $\xi_{t,1}^{\beta} = \xi_{t^{\beta},1}$.*

Proof. Suppose $p=3$ and $\phi(t)^\beta=\phi(t)$. Then $\phi(t)^{\beta^2}=\phi(t^\alpha)=\phi(t)$, so that $t^\alpha=t^3=t$ and $t^2=1$.

(a) Since $N(\phi(1))/T_\epsilon=M_\epsilon/T_\epsilon\simeq\mathbb{Z}_2\times S_3$, $\phi(1)$ has 6 extensions to M_ϵ. Suppose $p=3$. As shown in the proof of (3D) (3), $\langle M_\epsilon,\theta\rangle/T_\epsilon\simeq(\mathbb{Z}_4\times 3).2$ has 4 linear characters and 5 irreducible characters of degree 2, where $\theta=\beta'\langle\alpha\rangle$. Thus β stabilizes exactly 2 linear characters and the two characters of degree 2 in $\mathcal{S}(1)$. If $p\neq 3$, then α stabilizes each character of $\mathcal{S}(1)$, since $w^\alpha=w$ for all $w\in\langle\rho,\tau,\sigma\rangle$.

(b) The character $\phi(t)$ has four linear extensions to $N(\phi(t))=\langle T_\epsilon,\rho,\tau\sigma\rangle$. If $p=3$, then β' stabilizes $N(\phi(t))$ and $\langle N(\phi(t)),\theta\rangle/T_\epsilon\simeq D_8$, so that β' stabilises exactly two linear characters of $\mathcal{S}(t)$. If $p\neq 3$, then the proof is similar to that of (a).

(c) The character $\phi(t)$ has three extensions $\xi_{t,i}$ to $N(\phi(t))=\langle T_\epsilon,\sigma,\tau\rangle$, and $\xi_{t,i}^\alpha=\xi_{t^\alpha,i}$.

The proofs of (d) and (e) are similar to (c). $\square$

(4C) *Let $G=G_2(q)$ and let r be a prime such that* $\gcd(r,q)=1$. *If $r\geqslant 3$, then Uno's invariant conjecture holds for G.*

Proof. Suppose $B\in\mathrm{Blk}^0(G,r)$ with a defect group D such that $B\subseteq\mathcal{E}_r(G,(s))$, and suppose (D,b_D) is a maximal B-pair. If $D=O_2(T_\epsilon)$, then by [7, Theorem 3.2], we may suppose $b_D=\mathcal{E}_r(T_\epsilon,(s))$.

(1) Let $C=C(1)$ or $C(1)^1$ according as $r=3$ or $r\geqslant 5$. Then $N(C)=K_\epsilon$ or $N(T_\epsilon)$, and $\mathrm{Blk}(N(C)|B)=\{b(C)\}$, where $\mathrm{Blk}(N(C)|B)=\{b\in\mathrm{Blk}(N(C))\colon\ b^G=B\}$. We will define a bijection

$$\Psi:\mathrm{Irr}\big(b(C)\big)\longrightarrow\mathrm{Irr}(B)\tag{4.3}$$

such that Ψ is an isomorphism of $\langle\beta\rangle$-sets, and $w(\Psi(\varphi))\equiv\pm w(\varphi)\pmod r$ for $\varphi\in\mathrm{Irr}(b(C))$.

(1.1) Suppose $r\geqslant 5$. Let $Q\leqslant G$, and $\mathcal{Y}(b(C),Q)=\{I(\xi)\colon\ \xi\in\mathcal{S}(t)\}$, where $I(\xi)=\mathrm{Ind}_{N(\phi(t))}^{N(C)}(\xi)$ and t runs over T_ϵ such that $\phi(t)\in\mathrm{Irr}(b_D)$ and $C_G(t)=Q$. Define the map

$$\Psi:\mathcal{Y}\big(b(C),Q\big)\longrightarrow\mathcal{Y}(B,Q)\tag{4.4}$$

as follows:

If $C_G(t)=T_\epsilon$, then $\mathcal{E}(G,(t))=\{\chi_{t,1}\}$ and define $\Psi(I(\xi_{t,1}))=\chi_{t,1}$.

If $C_G(t)\simeq J_\epsilon$, then $\mathcal{E}(G,(t))=\{\chi_{t,1},\chi_{t,\mathrm{St}}\}$ and define $\Psi(I(\xi_{t,1}))=\chi_{t,1}$ and $\Psi(I(\xi_{t,2}))=\chi_{t,\mathrm{St}}$, where St is the Steinberg character of $C_G(t)$.

If $C_G(t)=L_\epsilon$, then $\mathcal{E}(G,(t))=\{X_{31},X_{32},X_{33}\}$ (in the notation of [19, Appendix B]), and we define $\Psi(I(\xi_{t,i}))=X_{3i}$ for $\xi_{t,i}\in\mathcal{S}(t)$.

If $|t|=2$, then $\mathcal{E}(G,(t))=\{X_{21},X_{22},X_{23},X_{24}\}$, and we define $\Psi(I(\xi_{t,i}))=X_{2i}$ for $\xi_{t,i}\in\mathcal{S}(t)$.

If $t=1$, then $\mathcal{E}(G,(t))\cap\mathrm{Irr}(B_1)=\{X_{11},X_{12},X_{13},X_{14},X_{1\gamma_1},X_{1\gamma_2}\}$, where $(\gamma_1,\gamma_2)=(5,6)$ or $(7,8)$ according as $\epsilon=+$ or $-$. Define $\Psi(I(\xi_{1,i}))=X_{1i}$ for $\xi_{1,i}\in\mathcal{S}(1)$ except when $\epsilon=-$ and $i=5,6$, in which case $\Psi(I(\xi_{1,i}))=X_{1i+2}$.

Then Ψ induces a bijection between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B)$ (see [19]). By (4A) and (4B), $\omega\in\langle\beta\rangle$ stabilizes $\Psi(I(\xi))$ if and only if ω stabilizes $I(\xi)$ for $\xi\in\mathcal{S}(t)$, since

$$\mathrm{Ind}_{N(\phi(t))}^{N(C)}(\xi)^\beta=\mathrm{Ind}_{N(\phi(t))^\beta}^{N(C)}\big(\xi^\beta\big).$$

Suppose $\chi_{t,\mu} \in \mathrm{Irr}(G)$. Since $r | q-\epsilon$, it follows that

$$w(\chi_{t,\mu}) \equiv \frac{|G|_p}{|C_G(t)|_p} w(\mu) \equiv \pm w(\mu) \pmod r. \tag{4.5}$$

In addition, if $K \leqslant H$ and $\xi \in \mathrm{Irr}(K)$, then

$$w\big(\mathrm{Ind}_K^H(\xi)\big) \equiv w(\xi) \pmod r. \tag{4.6}$$

It follows by (4.5) and (4.6) that $w(\Psi(\varphi)) \equiv \pm w(\varphi) \pmod r$ for any $\varphi \in \mathrm{Irr}(b(C))$, so that

$$\mathrm{k}\big(G, B, d, u, [w]\big) = \mathrm{k}\big(N(C), B, d, u, [w]\big) \tag{4.7}$$

for all $d, u, w \in \mathbb{Z}$.

(1.2) Suppose $r = 3$ and $B = B_1$. Then $p \neq 3$, $N(C) = K_\epsilon = \langle L_\epsilon, \rho\rangle$ and $b(C) = B_0(K_\epsilon)$. Let $b_0 = B_0(L_\epsilon)$ and $B_H = B_0(H)$, where $L_\epsilon \leqslant H = \mathrm{GL}_3^\epsilon(q)$. Then B_H covers b_0, $B_H = \mathcal{E}_3(H, (1))$ by [15] and $b_0 = \mathcal{E}_3(L_\epsilon, (1))$. Let $\zeta_{t,\mu}$ be the irreducible character of H labelled by (t, μ) and $\zeta_{t,\mu}|_{L_\epsilon}$ the restriction of $\zeta_{t,\mu}$ to L_ϵ, where $t \in L_\epsilon$ is semisimple and μ is a unipotent character of $C_H(t)$.

In the notation of [25], suppose $\xi_i = \chi^{(i)}_{(q-\epsilon)^2(q+\epsilon)/3}$ for $i = 0, 1, 2$. (In the notation of [18], $\xi_i = \chi^{(i,1)}_{(q-\epsilon)^2(q+\epsilon)/3}$.) Then the ξ_i are the three irreducible constituents of $\zeta_{y,1}|_{L_\epsilon}$ for some 3-element $y \in H$ such that $C_H(y) \simeq \mathbb{Z}_{q^3-1}$. Since y and y^ρ are not H-conjugate, it follows that $\zeta_{y,1} \neq \zeta_{y^\rho,1}$ and $\xi_i^\rho \neq \xi_i$ for all i, where ρ is extended as the inverse-transpose map on H. Thus $\mathrm{Irr}^{2a-1}(b_0) = \{\xi_i, \xi_i^\rho \colon i = 0, 1, 2\}$.

In the notation of [25], α stabilizes the class $C_3^{(0,0)}$, so $\xi_0^\alpha = \xi_0$, and $\xi_1^\alpha = \xi_2$ if and only if $(C_3^{(0,1)})^\alpha = C_3^{(0,2)}$. By [20, Tables I and II], $\mathrm{Irr}^{2a-1}(B_1) = \{X_{19}, \overline{X}_{19}, X_{18}\}$ or $\mathrm{Irr}^{2a-1}(B_1) = \{X_{19}, \overline{X}_{19}, X_{16}\}$ according as $\epsilon = +$ or $-$. If $p = 2$, then in the notation of [13], α stabilizes each class B_0, B_1 and $B_2(0)$, and $B_2(1)^\alpha = B_2(2)$ if and only if $(C_3^{(0,1)})^\alpha = C_3^{(0,2)}$, and by [13, Tables IV-2 and IV-3], $B_2(1)^\alpha = B_2(2)$ if and only if $X_{19}^\alpha = \overline{X}_{19}$. If $p \geqslant 5$, then in the notation of [9], α stabilizes each class k_3, $k_{3,1}$ and $k_{3,2}$, and $k_{3,3,1}^\alpha = k_{3,3,2}$ if and only if $(C_3^{(0,1)})^\alpha = C_3^{(0,2)}$, if and only if $X_{19}^\alpha = \overline{X}_{19}$. It follows that $\xi_1^\alpha = \xi_2$ if and only if $X_{19}^\alpha = \overline{X}_{19}$. Define

$$\Psi : \mathrm{Irr}^{2a-1}\big(b(C)\big) \longrightarrow \mathrm{Irr}^{2a-1}(B_1)$$

such that $\Psi(\xi_0 + \xi_0^\rho) = X_{18}$ or X_{16} according as $\epsilon = +$ or $-$, $\Psi(\xi_1 + \xi_1^\rho) = X_{19}$ and $\Psi(\xi_2 + \xi_2^\rho) = \overline{X}_{19}$. Then $\omega \in \langle\beta\rangle$ stabilizes $\Psi(\varphi)$ if and only if it stabilizes φ.

In the notation of [25] and [17],

$$\mathrm{Irr}^0(b_0) = \big\{\chi_1, \chi_{q^2+\epsilon q}, \chi_{q^3}, \chi^{(i)}_{(q+\epsilon)(q^2+\epsilon q+1)/3} \colon 0 \leqslant i \leqslant 2\big\}.$$

In addition, the $\eta_i = \chi^{(i)}_{(q+\epsilon)(q^2+\epsilon q+1)/3}$ are the three irreducible constituents of $\zeta_{t,1}$ for some 3-element $t \in L_\epsilon$ such that $C_H(t) \simeq \mathbb{Z}_{q-\epsilon} \times \mathbb{Z}_{q-\epsilon} \times \mathbb{Z}_{q-\epsilon}$. Since η_i^ρ are constituents of $\zeta_{t,1}^\rho = \zeta_{t^{-1},1}$, it follows that $\zeta_{t,1}|_{L_\epsilon} = \zeta_{t^{-1},1}|_{L_\epsilon}$ and we may suppose $t = \mathrm{diag}\{t_1, t_1^{-1}, 1\} \in T_\epsilon$ for some element $1 \neq t_1 \in O_3(\mathbb{Z}_{q-\epsilon})$. Since $\langle H, \rho\rangle / Z(H)L_\epsilon \simeq S_3$, ρ stabilizes exactly one constituent η_0. Thus ρ

stabilizes exactly 4 characters of $\mathrm{Irr}^0(b_0)$, so that $\mathrm{Irr}^0(b(C))$ has 9 characters. Since $\rho^\alpha = \rho$, it follows that α stabilizes each character of $\mathrm{Irr}^0(b(C))$. Moreover, $|\mathrm{Irr}^0(B_1)| = 9$ and α stabilizes each of them.

Now we consider $\mathrm{Irr}^1(b(C))$ and $\mathrm{Irr}^1(B_1)$. As shown above L_ϵ has exactly one class of non-trivial 3-elements $t = \mathrm{diag}\{t_1, t_1^{-1}, 1\}$ such that $C_{L_\epsilon}(t) = T_\epsilon$ and $\zeta_{t,1}|_{L_\epsilon}$ is reducible with constituents η_i. Let

$$\mathcal{X}(b_0) = \mathcal{Y}(b_0, T_\epsilon)\backslash\{\eta_0, \eta_1, \eta_2\}.$$

By [25, Table 1b] and [17, Theorem 4.5], $|\mathcal{X}(b_0)| = \frac{1}{6}3^a(3^a - 3)$. Let

$$\mathcal{X}_0(b_0) = \{\xi \in \mathcal{X}(b_0)\colon\ \xi^\rho = \xi\}, \qquad \mathcal{X}_1(b_0) = \mathcal{X}(b_0)\backslash\mathcal{X}_0(b_0).$$

If $\xi_y = \zeta_{y,1}|_{L_\epsilon} \in \mathcal{X}_0(b_0)$, then we may suppose $y = \mathrm{diag}\{y_1, y_1^{-1}, 1\}$ for some $y_1 \in O_3(\mathbb{Z}_{q-\epsilon})$, so that $C_G(y) \simeq J_\epsilon$. Thus $|\mathcal{X}_0(b_0)| = \frac{1}{2}(3^a - 3)$ and $|\mathcal{X}_1(b_0)| = \frac{1}{6}3^a(3^a - 3) - \frac{1}{2}(3^a - 3)$. Let

$$\mathcal{X}_1(C) = \{\varphi_{y,1} = \mathrm{Ind}_{L_\epsilon}^{K_\epsilon}(\xi_y)\colon\ \xi_y \in \mathcal{X}_1(b_0)\}$$

and $\mathcal{X}_0(C) = \{\varphi_{y,i} \in \mathrm{Irr}(K_\epsilon)\colon\ \varphi_{y,i}$ cover $\xi_y \in \mathcal{X}_0(b_0),\ i = 1, 2\}$. Then $|\mathcal{X}_0(C)| = 3^a - 3$ and $|\mathcal{X}_1(C)| = \frac{1}{12}(3^a - 3)^2$.

Let $\mathcal{X}_0(B_1) = \{\{X_\gamma\}, \{X'_\gamma\}\} \subseteq \mathcal{Y}(B_1, J_\epsilon)$ and $\mathcal{X}_1(B_1) = \{\{X_\lambda\}\} = \mathcal{Y}(B_1, T_\epsilon)$, where $(\gamma, \lambda) = (1a, 1)$ or $(2b, 2)$ according as $\epsilon = +$ or $-$. Then $|\mathcal{X}_i(B_1)| = |\mathcal{X}_i(C)|$ for $i = 0, 1$, $\mathcal{X}_0(B_1) = \{\chi_{y,1}, \chi_{y,\mathrm{St}}\}$ with $y = \mathrm{diag}\{y_1, y_1^{-1}, 1\} \in O_3(T_\epsilon)$ and $\mathcal{X}_1(B_1) = \{\chi_{y,1}\}$ with $C_G(y) = T_\epsilon$. Define $\Psi : \mathcal{X}_i(C) \to \mathcal{X}_i(B_1)$ by $\Psi(\varphi_{y,1}) = \chi_{y,1}$ and $\Psi(\varphi_{y,2}) = \chi_{y,\mathrm{St}}$. Then Ψ is a required bijection between $\mathcal{X}_i(C)$ and $\mathcal{X}_i(B_1)$.

If $\zeta_{y,\mu} \in \mathcal{Y}(B_H, J_\epsilon \times \mathbb{Z}_{q-\epsilon})$ with $y \in L_\epsilon$, then $\xi_{y,\mu} = \zeta_{y,\mu}|_{L_\epsilon}$ is irreducible and $\xi_{y,\mu} \in \mathcal{X}_2(b_0) = \mathcal{Y}(b_0, J_\epsilon)$. Thus $\mathcal{X}_2(b_0) = \{\xi_{y,1}, \xi_{y,\mathrm{St}}\colon\ C(y) = J_\epsilon\}$ and $|\mathcal{X}_2(b_0)| = 2(3^a - 1)$. Since $\zeta_{y,1}^\rho = \zeta_{y^{-1},1}$ and $\zeta_{y,\mathrm{St}}^\rho = \zeta_{y^{-1},\mathrm{St}}$, it follows that ρ stabilizes no characters of $\mathcal{X}_2(b_0)$. Let

$$\mathcal{X}_2(C) = \{\varphi_{y,\mu} = \mathrm{Ind}_{L_\epsilon}^{K_\epsilon}(\xi_{y,\mu})\colon\ \xi_{y,\mu} \in \mathcal{X}_2(b_0)\}$$

and $\mathcal{X}_2(B_1) = \mathcal{Y}(B_1, J_\epsilon)\backslash\mathcal{X}_0(B_1)$. Then

$$\mathcal{X}_2(B_1) = \{\chi_{y,\mu}\colon\ y \in O_3(T_\epsilon),\ C_{L_\epsilon}(y) = J_\epsilon,\ \mu = 1, \mathrm{St}\}$$

and $|\mathcal{X}_2(C)| = |\mathcal{X}_2(B_1)| = 3^a - 1$. Thus $\Psi(\varphi_{y,1}) = \chi_{y,1}$ is a required bijection between $\mathcal{X}_2(C)$ and $\mathcal{X}_2(B_1)$.

Since $\mathrm{Irr}^1(b_0) = \bigcup_{i=0}^2 \mathcal{X}_i(b_0)$, it follows that $\mathrm{Irr}^1(b(C)) = \bigcup_{i=0}^2 \mathcal{X}_i(C)$ and by [20, Tables 1 and 2], Ψ is a bijection between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B_1)$.

(1.3) Suppose $r = 3$ and $B \neq B_1$. Then the proof of (4.7) is similar to the proof (1.2) above, so we sketch a proof.

Let $b = b_D^{L_\epsilon}$, so that $b = \mathcal{E}_r(L_\epsilon, (s))$. If $B = B_a$ or B_b, then $C_{L_\epsilon}(s) = J_\epsilon$,

$$\mathcal{Y}(b, J_\epsilon) = \{\xi_{sy,\mu}\colon\ y \in O_3(T_\epsilon),\ C_{L_\epsilon}(sy) = J_\epsilon,\ \mu = 1, \mathrm{St}\},$$

and $\mathcal{Y}(b, T_\epsilon) = \{\xi_{sy,1}\colon\ y \in O_3(T_\epsilon),\ C_{L_\epsilon}(sy) = T_\epsilon\}$, where $\xi_{sy,\mu} = \zeta_{sy,\mu}|_{L_\epsilon}$. Let

$$\mathcal{Y}(C, Q) = \bigl\{\varphi_{y,\mu} = \mathrm{Ind}_{L_\epsilon}^{K_\epsilon}(\xi_{y,\mu})\colon\ \xi_{t,\mu} \in \mathcal{Y}(b, Q)\bigr\}$$

for $Q \in \{J_\epsilon, T_\epsilon\}$. Then $|\mathcal{Y}(b, J_\epsilon)| = |\mathcal{Y}(C, J_\epsilon)| = 2 \cdot 3^a$ and $|\mathcal{Y}(b, T_\epsilon)| = |\mathcal{Y}(C, T_\epsilon)| = \frac{1}{2}(3^{2a} - 3^a)$. In addition, $\varphi_{y,\mu}^\alpha = \varphi_{y^\alpha,\mu}$.

By [20, pp. 374 and 377], $|\mathcal{Y}(B, J_\epsilon)| = 2 \cdot 3^a$ and $|\mathcal{Y}(B, T_\epsilon)| = \frac{1}{2}(3^{2a} - 3^a)$. Thus the map $\Psi(\varphi_{y,\mu}) = \chi_{y,\mu}$ is a required bijection between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B)$.

Suppose $B = B_{X_\gamma}$, where $\gamma = 1$ or 2 according as $\epsilon = +$ or $-$. Then

$$\mathrm{Irr}\bigl(b(C)\bigr) = \bigl\{\varphi_{sy,1} = \mathrm{Ind}_{L_\epsilon}^{K_\epsilon}(\xi_{sy,1})\colon\ y \in O_3(T_\epsilon),\ \xi_{sy,1} \in \mathcal{Y}(b, T_\epsilon)\bigr\}$$

and $|\mathrm{Irr}(b(C))| = |\mathrm{Irr}(B)| = |\mathrm{Irr}(b)| = 3^{2a}$. Thus $\Psi(\varphi_{sy,1}) = \chi_{sy,1}$ is a required bijection between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B)$

Finally, suppose $B = B_2$, so that $|s| = 2$, $C_{L_\epsilon}(sy) = J_\epsilon$ or T_ϵ for $y \in T_\epsilon$. We may suppose $s = h_1(-1) = \mathrm{diag}\{-1, -1, 1\} \in L_\epsilon$. If $\xi_{sy,\mu} \in \mathcal{Y}(b, J_\epsilon)$ is stabilized by ρ, then $y = 1$ and let $\varphi_{s,j}$ for $1 \leqslant j \leqslant 4$ be the 4 extensions of $\xi_{s,\mu}$ in K_ϵ, where $\mu = 1, \mathrm{St}$. Let $\mathcal{X}_0(b)$ be the characters of $\mathcal{Y}(b, T_\epsilon)$ stabilized by ρ, $\mathcal{X}_1(b) = \mathcal{Y}(b, J_\epsilon)\backslash\{\xi_{s,1}, \xi_{s,\mathrm{St}}\}$ and $\mathcal{X}_2(b) = \mathcal{Y}(b, T_\epsilon)\backslash\mathcal{X}_0(b)$. If $\xi_{sy,1} \in \mathcal{X}_0(b)$, then we may suppose $sy = \mathrm{diag}\{-c, -c^{-1}, 1\} \in O_3(T_\epsilon)\backslash\{1\}$. It follows that

$$\bigl|\mathcal{X}_i(b)\bigr| = \begin{cases} \frac{1}{2}(3^a - 1) & \text{if } i = 0,\\ 2(3^a - 1) & \text{if } i = 1,\\ \frac{1}{2}(3^{2a} - 3^a - (3^a - 1)) & \text{if } i = 2.\end{cases}$$

Let $\mathcal{X}_i(C) = \{\varphi_{y,\mu} = \mathrm{Ind}_{L_\epsilon}^{K_\epsilon}(\xi_{y,\mu})\colon\ \xi_{y,\mu} \in \mathcal{X}_i(b)\}$ for $i = 1, 2$. In addition, let $\varphi_{sy,1}$ and $\varphi_{sy,\mathrm{St}}$ be the two extensions of $\xi_{sy,1} \in \mathcal{X}_0(b)$ to K_ϵ and

$$\mathcal{X}_0(C) = \bigl\{\varphi_{sy,\mu}, \varphi_{s,j}\colon\ \xi_{sy,1} \in \mathcal{X}_0(b),\ 1 \leqslant j \leqslant 4\bigr\}.$$

Then

$$\bigl|\mathcal{X}_i(C)\bigr| = \begin{cases} 3 + 3^a & \text{if } i = 0,\\ 3^a - 1 & \text{if } i = 1,\\ \frac{1}{4}(3^a - 1)^2 & \text{if } i = 2.\end{cases}$$

Let $\mathcal{X}_0(B) = \{\chi_{sy,\mu}\colon\ y = \mathrm{diag}\{-c, -c^{-1}, 1\} \in O_3(T_\epsilon)\}$, $\mathcal{X}_1(B) = \mathcal{Y}(B, J_\epsilon)\backslash\mathcal{X}_0(B)$ and $\mathcal{X}_2(B) = \mathcal{Y}(B, T_\epsilon)$. By [20], $|\mathcal{X}_i(B)| = |\mathcal{X}_i(C)|$ for $i = 0, 1, 2$. Given $i = 1, 2$, define $\Psi : \mathcal{X}_i(C) \to \mathcal{X}_i(B)$ by $\Psi(\varphi_{sy,\mu}) = \chi_{sy,\mu}$. In addition, define $\Psi : \mathcal{X}_0(C) \to \mathcal{X}_0(B)$ by $\Psi(\varphi_{sy,\mu}) = \chi_{sy,\mu}$ except when $y = 1$, in which case $\Psi(\varphi_{s,j}) = X_{2j}$. Then Ψ is a required bijection between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B)$, so that (4.7) holds.

(2) Let $C = C(2)_1$ and $C' = C(2)$. Then $N(C) = N_G({J_\epsilon}^*)$ and $N(C') = N_{N(C)}(T_\epsilon)$. Since $N_G(J_\epsilon) \simeq N_G({J_\epsilon}^*)$, for simplicity of notation, we identify J_ϵ with ${J_\epsilon}^*$ and suppose $Z(J_\epsilon)$ is generated by an element $\mathrm{diag}\{c^{-2}, c, c\} \in L_\epsilon$ for some $c \in \mathbb{Z}_{q-\epsilon}$. Thus $N(C) = \langle J_\epsilon, \rho\rangle$, $N(C') = \langle T_\epsilon, \tau, \rho\rangle$, $N_A(C) = N(C){:}\langle\alpha\rangle$ and $N_A(C') = \langle T_\epsilon, \tau, \rho, \alpha\rangle$.

The Brauer correspondence induces a bijection from $\mathrm{Blk}(N(C')|B)$ to $\mathrm{Blk}(N(C)|B)$. Let (D, b_D) be a maximal B-pair, $b(C') = b_D^{N(C')} \in \mathrm{Blk}(N(C')|B)$ and $b(C) = b_D^{N(C)} \in \mathrm{Blk}(N(C)|B)$.

Let $H_\epsilon = \langle T_\epsilon, \tau\rangle$, $B' = b_D^{J_\epsilon}$ and $b' = b_D^{H_\epsilon}$, and let $N(B')$ and $N(b')$ be stabilizers of B' and b' in $N(C)$ and $N(C')$, respectively. Then $H_\epsilon = N_{J_\epsilon}(D)$, $b'^{J_\epsilon} = B'$, $b'^{N(C')} = b(C')$ and $B'^{N(C)} = b(C)$. If $N(B') = J_\epsilon$, then $N(b') = H_\epsilon$. By [14, Theorem V.2.5], $\mathrm{Ind}_{J_\epsilon}^{N(C)}$ induces a bijection of $\mathrm{Irr}(B')$ onto $\mathrm{Irr}(b(C))$. Since $[\alpha, \rho] = 1$, it follows that for $\varphi \in \mathrm{Irr}(B')$,

$$\left(\mathrm{Ind}_{J_\epsilon}^{N(C)}(\varphi)\right)^\alpha = \mathrm{Ind}_{J_\epsilon}^{N(C)}\left(\varphi^\alpha\right),$$

so that $\mathrm{Irr}(B') \simeq \mathrm{Irr}(b(C))$ as $\langle\alpha\rangle$-sets and by (4.6), $w(\mathrm{Ind}_{J_\epsilon}^{N(C)}(\varphi)) \equiv w(\varphi) \pmod r$.

Similarly, $\mathrm{Ind}_{H_\epsilon}^{N(C')}$ induces a bijection of $\mathrm{Irr}(b')$ onto $\mathrm{Irr}(b(C'))$, $\mathrm{Irr}(b') \simeq \mathrm{Irr}(b(C'))$ as $\langle\alpha\rangle$-sets and $w(\mathrm{Ind}_{H_\epsilon}^{N(C')}(\varphi)) \equiv w(\varphi) \pmod r$ for $\varphi \in \mathrm{Irr}(b')$. But $\mathrm{k}(B') = \mathrm{k}(B', 0) = \mathrm{k}(b') = \mathrm{k}(b', 0)$, so

$$\mathrm{k}\big(b(C), d, u, [w]\big) = \mathrm{k}\big(B', d, u, [w]\big) = \mathrm{k}\big(b', d, u, [w]\big) = \mathrm{k}\big(b(C'), d, u, [w]\big),$$

for all integers d, u, r, where $\mathrm{k}(b(C), d, u, [w]) = \mathrm{k}(N(C), B, d, u, [w]) \cap \mathrm{Irr}(b(C))$.

Suppose $N(B') = N(C)$, so that $N(b') = N(C')$. If N is a subgroup of G containing T_ϵ, we denote by $[T_\epsilon, \phi(t)]_N$ the N-orbit containing the pair $(T_\epsilon, \phi(t))$. In particular, $N(C)$ and $N(C')$ permute $\{[T_\epsilon, \phi(t)]_{J_\epsilon}\}$ and $\{[T_\epsilon, \phi(t)]_{H_\epsilon}\}$, respectively. By [7, Theorem 3.2], $\mathrm{Irr}(B') = \mathcal{E}_r(J_\epsilon, (s))$.

If $\chi \in \mathcal{Y}(B', T_\epsilon)$, then $\chi = \pm R_{T_\epsilon}^{J_\epsilon}(\phi(sy))$ for some sign $\pm$, and there is a bijection of $\mathcal{Y}(B', T_\epsilon)$ onto the set $\mathcal{X}(J_\epsilon) = \{[T_\epsilon, \phi(sy)]_{J_\epsilon}\colon y \in D,\ C_{J_\epsilon}(sy) = T_\epsilon\}$. The inclusion map $[T_\epsilon, \phi(sy)]_{H_\epsilon} \subseteq [T_\epsilon, \phi(sy)]_{J_\epsilon}$ is a bijection of

$$\mathcal{X}(H_\epsilon) = \left\{\left[T_\epsilon, \phi(sy)\right]_{H_\epsilon}\colon\ y \in D,\ C_{H_\epsilon}(sy) = T_\epsilon\right\}$$

onto $\mathcal{X}(J_\epsilon)$. Let $\mathcal{Y}(b', T_\epsilon)$ be the subset of $\mathrm{Irr}(b')$ consisting of characters $\zeta(sy)$ of H_ϵ covering $\phi(sy)$ with $[T_\epsilon, \phi(sy)]_{H_\epsilon} \in \mathcal{X}(H_\epsilon)$. If $\zeta(sy) \in \mathcal{Y}(b', T_\epsilon)$, then $\phi(sy)^\tau \neq \phi(sy)$, since $C_{J_\epsilon}(sy) = T_\epsilon$. Thus $\zeta(sy) \mapsto [T_\epsilon, \phi(sy)]_{H_\epsilon}$ is a bijection of $\mathcal{Y}(b', T_\epsilon)$ onto $\mathcal{X}(H_\epsilon)$. The map $\Psi_1(\zeta(sy)) = \pm R_{T_\epsilon}^{J_\epsilon}(\phi(sy))$ for suitable sign $\pm$ is an isomorphism of $\mathcal{Y}(b', T_\epsilon)$ and $\mathcal{Y}(B', T_\epsilon)$ as $\langle\alpha\rangle$-sets. In addition, by (4.5),

$$w\big(\Psi_1\big(\zeta(sy)\big)\big) \equiv \pm|T_\epsilon|_{r'} \equiv \pm w\big(\zeta(sy)\big) \ (\mathrm{mod}\ r). \tag{4.8}$$

Next suppose $\varphi \in \mathrm{Irr}(b(C))$ and $\psi \in \mathrm{Irr}(b(C'))$ covering $\xi = \Psi_1(\zeta(sy))$ and $\zeta(sy)$, respectively. If $\xi^\rho = \xi$, then $[T_\epsilon, \phi(sy)]_{J_\epsilon} = [T_\epsilon, \phi(sy)^{-1}]_{J_\epsilon}$, and $\varphi^\omega = \varphi$ if and only if $\xi^\omega = \xi$ for $\omega \in \langle\alpha\rangle$. Since $[T_\epsilon, \phi(sy)]_{H_\epsilon} = [T_\epsilon, \phi(sy)^{-1}]_{H_\epsilon}$, it follows that ρ stabilizes $\zeta(sy)$, so that ω stabilizes ψ if and only if it stabilizes $\zeta(sy)$.

If $\xi^\rho \neq \xi$, then $[T_\epsilon, \phi(sy)]_{J_\epsilon} \neq [T_\epsilon, \phi(sy)^{-1}]_{J_\epsilon}$ and $\varphi|_{J_\epsilon} = \xi + \xi^\rho$. Thus $[T_\epsilon, \phi(sy)]_{H_\epsilon} \neq [T_\epsilon, \phi(sy)^{-1}]_{H_\epsilon}$ and $\psi = \zeta(sy) + \zeta(sy)^\rho$. In addition, $\omega \in \langle\alpha\rangle$ stabilizes φ if and only if it stabilizes the set $\{[T_\epsilon, \phi(sy)]_{J_\epsilon}, [T_\epsilon, \phi(sy)^{-1}]_{J_\epsilon}\}$, if and only if it stabilizes the set

$$\left\{\left[T_\epsilon, \phi(sy)\right]_{H_\epsilon}, \left[T_\epsilon, \phi(sy)^{-1}\right]_{H_\epsilon}\right\},$$

if and only if ω stabilizes ψ. It follows by (4.8) that $w(\varphi) \equiv \pm w(\psi) \pmod r$.

Let $\mathcal{Y}(b(C), T_\epsilon) \subseteq \mathrm{Irr}(b(C))$ and $\mathcal{Y}(b(C'), T_\epsilon) \subseteq \mathrm{Irr}(b(C'))$ be the subsets of characters covering $\mathcal{Y}(B', T_\epsilon)$ and $\mathcal{Y}(b', T_\epsilon)$, respectively. Then there is a required bijection Ψ between $\mathcal{Y}(b(C), T_\epsilon)$ and $\mathcal{Y}(b(C'), T_\epsilon)$.

If $\xi_{sy,\mu} \in \mathcal{Y}(B', J_\epsilon)$, then $sy \in Z(J_\epsilon)$ and $\phi(sy)^\tau = \phi(sy)$. So $\mathrm{Irr}(b')$ contains two characters $\zeta_{sy,1}$ and $\zeta_{sy,2}$ covering $\phi(sy)$. Define $\Psi_1(\xi_{sy,1}) = \zeta_{sy,1}$ and $\Psi_1(\xi_{sy,\mathrm{St}}) = \zeta_{sy,2}$, so that $w(\Psi_1(\xi_{sy,\mu})) \equiv \pm w(\xi_{sy,\mu}) \pmod r$. By [15, (2H)], $\xi_{sy,\mu}^\rho = \xi_{sy,\mu}$ if and only $\phi(sy)^\rho = \phi(sy)$, if and only if $\zeta_{sy,i}^\rho = \zeta_{sy,i}$. If ρ stabilizes $\xi_{sy,\mu}$, then $\mathrm{Irr}(b(C))$ has two characters $\varphi_{sy,\mu}$ and $\varphi'_{sy,\mu}$ covering $\xi_{sy,\mu}$, and $\mathrm{Irr}(b(C'))$ has two characters $\psi_{sy,i}$ and $\psi'_{sy,i}$ covering $\zeta_{sy,i}$. There is a required bijection Ψ between $\{\varphi_{sy,\mu}, \varphi'_{sy,\mu}\colon \mu = 1, \mathrm{St}\}$ and $\{\psi_{sy,i}, \psi'_{sy,i}\colon i = 1, 2\}$. If $\xi_{sy,\mu}^\rho \neq \xi_{sy,\mu}$, then $\zeta_{sy,i}^\rho \neq \zeta_{sy,i}$, and so $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(b(C'))$ each has exactly one character $\varphi_{sy,\mu}$ and $\psi_{sy,i}$ covering $\xi_{sy,\mu}$ and $\zeta_{sy,i}$, respectively. We define $\Psi(\varphi_{sy,1}) = \psi_{sy,1}$ and $\Psi(\varphi_{sy,\mathrm{St}}) = \psi_{sy,2}$. Thus Ψ is a required bijection between $\mathrm{Irr}(b(C)) \backslash \mathcal{Y}(b(C), T_\epsilon)$ and $\mathrm{Irr}(b(C')) \backslash \mathcal{Y}(b(C'), T_\epsilon)$. It follows that for $d, u, w \in \mathbb{Z}$

$$\mathrm{k}\big(N(C), B, d, u, [w]\big) = \mathrm{k}\big(N(C'), B, d, u, [w]\big). \tag{4.9}$$

If $r \geqslant 5$, let $C = C(1)_1$ and $C' = C(1)$. The same proof as above shows that (4.9) still holds. $\square$

Remarks. (a) If $r \geqslant 5$, then (4.7) implies the Isaacs–Navarro conjecture for $G_2(q)$ since $N(C) = N(T_\epsilon)$ is also the normalizer of a defect group of B. If $r = 2$ or 3, then the Isaacs–Navarro conjecture is equivalent to the Alperin–McKay conjecture, which has an affirmative answer in this case.

(b) Suppose $p = r$. If χ is an irreducible character of $B_0(G_2(q))$ or the Borel subgroup, then $w(\chi) \equiv \pm 1 \pmod p$, so that the Isaacs–Navarro conjecture is equivalent to the Alperin–McKay conjecture, which also has an affirmative answer in this case.

5. The proof of the conjecture for the even prime

The notation and terminology of Sections 2 and 3 are continued in this section. Suppose $B \in \mathrm{Blk}^0(G, 2)$ with a defect group D, where $\mathrm{Blk}^0(G, 2)$ is given by (4.1). Then we may suppose $D \in \{O_2(T_\delta), SD_{2^{a+2}}, \mathbb{Z}_{2^a} \wr \mathbb{Z}_2, S\}$, where $S \in \mathrm{Syl}_2(G)$ and $SD_{2^{a+2}}$ is a semidihedral group of order 2^{a+2}. If $D \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ or $SD_{2^{a+2}}$, then Dade's invariant conjecture follows by [26, Theorem]. Let $\mathrm{Blk}^+(G, 2)$ be the blocks $B \in \mathrm{Blk}^0(G, 2)$ such that $D \in \{O_2(T_\epsilon), \mathbb{Z}_{2^a} \wr \mathbb{Z}_2, S\}$ and $B \in \mathrm{Blk}^+(G, 2)$. In addition, let $\mathcal{R}^0(B)$ be the subfamily of $\mathcal{R}^0(G)$ consisting of the chains C such that $\mathrm{Blk}(N(C)|B) \neq \emptyset$. By [3, (4A)],

$$\mathcal{R}^0(B)/G = \begin{cases} \{1, 1 < \mathbb{Z}_2, 1 < O_2(T_\epsilon), 1 < \mathbb{Z}_2 < O_2(T_\epsilon)\} & \text{if } D(B) \simeq O_2(T_\epsilon), \\ \{1, 1 < \mathbb{Z}_2, 1 < O_2(T_\epsilon), 1 < \mathbb{Z}_2 < O_2(T_\epsilon)\} & \text{if } D(B) \simeq \mathbb{Z}_{2^a} \wr \mathbb{Z}_2, \\ \mathcal{R}^0(G)/G & \text{if } B = B_1. \end{cases}$$

If $q = 3$, then $1 \neq D \in \Phi(G, 2)$, $C_G(D) = Z(D)$ and $B = B_0(G)$.

(5A) *Let q be a power of an odd prime p and B a non-principal 2-block of $G = G_2(q)$. Then B satisfies Dade's invariant conjecture.*

Proof. We may suppose $B \in \mathrm{Blk}^+(G,2)$ with a defect group D and $B \subseteq \mathcal{E}_2(G,(s))$ for some semisimple $2'$-element $s \in G$.

(1) Suppose D is abelian, so that $D = O_2(T_\epsilon)$ and $C_G(s) = T_\epsilon$. Let C: $1 < \mathbb{Z}_2$, C': $1 < \mathbb{Z}_2 < O_2(T_\epsilon)$, $b(C) \in \mathrm{Blk}(B|N(C))$ and $b(C') \in \mathrm{Blk}(B|N(C'))$. Then $N(C) = \mathrm{SO}(U_4)$ and $N(C') = \langle T_\epsilon, \rho, \tau\sigma\rangle$. The Brauer correspondence induces a bijection of $\mathrm{Blk}(N(C)|B)$ onto $\mathrm{Blk}(N(C')|B)$, we suppose $b(C')^{N(C)} = b(C)$.

By [7, Theorem 3.2] and [6, Theorem 2.3], $b(C) = \mathcal{E}_2(N(C),(s))$, $b_D^{N(C)} = b(C)$ and $b_D^{N(C')} = b(C')$, where $b_D = \mathcal{E}_2(T_\epsilon,(s))$. By (3.4), the stabilizer $N'(b_D)$ of b_D in $N(C')$ is T_ϵ, so that $N'(\xi) = T_\epsilon$ for all $\xi \in \mathrm{Irr}(b_D)$. Thus $\mathrm{Irr}(b(C')) = \{\mathrm{Ind}_{T_\epsilon}^{N(C')}(\phi(sy)) : y \in O_2(T_\epsilon)\}$ and

$$\mathrm{Irr}\big(b(C)\big) = \big\{\pm R_{T_\epsilon}^{N(C)}\big(\phi(sy)\big)\colon\ y \in O_2(T_\epsilon)\big\}.$$

So $\Psi(\mathrm{Ind}_{T_\epsilon}^{N(C')}(\phi(sy))) = \pm R_{T_\epsilon}^{N(C)}(\phi(sy))$ is an isomorphism of $\mathrm{Irr}(b(C'))$ and $\mathrm{Irr}(b(C))$ as $\langle\beta\rangle$-sets. If $d, u \in \mathbb{Z}$, then

$$\mathrm{k}\big(N(C'), B, d, u\big) = \mathrm{k}\big(N(C), B, d, u\big). \tag{5.1}$$

If C: $1 < O_2(T_\epsilon)$ and $b(C) \in \mathrm{Blk}(N(C)|B)$, then $N(C) = N_G(T_\epsilon)$ and the bijection Ψ defined by (4.4) induces a required bijection of $\mathrm{Irr}(b(C))$ onto $\mathrm{Irr}(B)$. Thus (5A) holds for B.

(2) Suppose $B \in \{B_{1a}, B_{1b}, B_{2a}, B_{2b}\}$. Since $D \simeq \mathbb{Z}_{2^a} \wr \mathbb{Z}_2$, it follows that $B \in \{B_{\gamma a}, B_{\gamma b}\}$ and $C_G(s) \simeq \mathrm{GL}_2^\epsilon(q)$, where $\gamma = 1$ or 2 according as $\epsilon = +$ or $-$.

Let $C = C(2)^1$ and $C' = C(2)$. Then $N(C) = N(T_\epsilon)$ and $N(C') = \langle T_\epsilon, \rho, \tau\sigma\rangle$. By [1, (2A)], the base subgroup $O_2(T_\epsilon)$ of $D \simeq \mathbb{Z}_{2^a} \wr \mathbb{Z}_2$ is the unique maximal normal abelian subgroup of D, so that $N(D) \leqslant N(O_2(T_\epsilon)) = N(C)$. Similarly, $N(D) \leqslant N(C')$ since $\Omega_1(Z(D)) = \mathbb{Z}_2$. Thus

$$\mathrm{Blk}\big(N(C)|B\big) = \big\{b(C)\big\}, \qquad \mathrm{Blk}\big(N(C')|B\big) = \big\{b(C')\big\}.$$

We may suppose $b^{N(C)} = b(C)$ and $b^{N(C')} = b(C')$, where $b = \mathcal{E}_2(T_\epsilon,(s))$. By (3.4), $\phi(s)^\sigma \neq \phi(s)$, and so $\sigma \notin N(b)$, where $N(b)$ is the stabilizer of b in $N(C)$. We may suppose $D \leqslant N(b)$ and $(N(b)\colon DT_\epsilon)_2 = 1$. Thus $N(\xi) = N'(\xi)$ and $(N(C) : N(\xi))_2 = (N(C') : N'(\xi))$ for $\xi \in \mathrm{Irr}(b)$, where $N'(\xi) = N(\xi) \cap N(C')$. The mappings $\mathrm{Ind}_{N(\xi)}^{N(C)}$ and $\mathrm{Ind}_{N'(\xi)}^{N(C')}$ induce a defect preserving bijection Ψ of $\mathrm{Irr}(b(C'))$ onto $\mathrm{Irr}(b(C))$. If $p = 3$, then $B^\beta \neq B$, so that $b(C)^\beta \neq b(C)$ and $b(C')^\beta \neq b(C')$. If $\xi = \phi(t)$ for some $t \in T_\epsilon$, then $\mathrm{Ind}_{N(\xi)}^{N(C)}(\zeta_t)^\alpha = \mathrm{Ind}_{N(\xi)}^{N(C)}(\zeta_{t^\alpha})$, where ζ_t is an extension of $\phi(t)$ to $N(\phi(t))$. Similarly, $\mathrm{Ind}_{N'(\xi)}^{N(C')}(\zeta_t)^\alpha = \mathrm{Ind}_{N'(\xi)}^{N(C')}(\zeta_{t^\alpha})$. So Ψ is an isomorphism of $\langle\alpha\rangle$-sets, and (5.1) holds.

Suppose $C = C(1)_1$: $1 < \mathbb{Z}_2$ and $b(C) \in \mathrm{Blk}(N(C)|B)$, so that $b(C) = \mathcal{E}_2(N(C),(s))$. Since $L = C_G(s)$ is a regular subgroup of $N(C)$, it follows by Broué [6, Theorem 2.3] that $R_L^{N(C)}$ induces a perfect isometry between $b(C)$ and $b_L = \mathcal{E}_2(L,(1))$. If $\xi_{y,\mu} \in \mathrm{Irr}(b_L)$, then $\pm R_L^{N(C)}(\xi_{y,\mu}) \in \mathrm{Irr}(b(C))$ for some sign $\pm$ and $R_L^{N(C)}(\xi_{y,\mu})^\alpha = R_L^{N(C)}(\xi_{y^\alpha,\mu})$. Similarly, since L is also regular in G, R_L^G induces a perfect isometry between $B = \mathcal{E}_2(G,(s))$ and b_L, and $R_L^G(\xi_{y,\mu})^\alpha = R_L^G(\xi_{y^\alpha,\mu})$ for $\xi_{y,\mu} \in \mathrm{Irr}(b_L)$. It follows that $\Psi(\pm R_L^{N(C)}(\xi_{y,\mu})) = \pm R_L^G(\xi_{y,\mu})$ is an $\langle\alpha\rangle$-set isomorphism of $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B)$ preserving defect.

(3) Suppose $B = B_3$, so that $q \equiv \epsilon \pmod 3$. Let $M = N(1 < \mathbb{Z}_2) = \mathrm{SO}(U_4)$, $b_M \in \mathrm{Blk}(M|B)$, $B_\epsilon = B_{1a}$ or B_{2a} according as $\epsilon = +$ or $-$. As shown in (2) $b_M = \mathcal{E}_2(M,(s))$, $R_L^M : b_L \to b_M$

and $R_L^G : b_L \to B_\epsilon$ induce perfect isometries, and ω stabilizes $R_L^G(\xi)$ or $R_L^M(\xi)$ if and only if it stabilizes $\xi \in \mathrm{Irr}(b_L)$ for $\omega \in \langle\alpha\rangle$ and $\xi \in \mathrm{Irr}(b_L)$. Thus we may identify $\mathrm{Irr}(b_M)$ with $\mathrm{Irr}(B_\epsilon)$. By [21],

$$\mathrm{Irr}^j(B_\epsilon) = \begin{cases} \{\varphi_{sy,\mu}\colon\ C(sy) = J_\epsilon, \mu = 1, \mathrm{St}\} & \text{if } j = 0,\\ \{\varphi_{sy,1}\colon\ C(sy) \simeq \mathbb{Z}_{q^2-1}\} & \text{if } j = a,\\ \{\varphi_{sy,1}\colon\ C(sy) = T_\epsilon\} & \text{if } j = 1,\\ \emptyset & \text{otherwise,} \end{cases}$$

and moreover, $|\mathrm{Irr}^0(B_\epsilon)| = 2^{a+1}$, $|\mathrm{Irr}^a(B_\epsilon)| = 2^{a-1}$ and $|\mathrm{Irr}^1(B_\epsilon)| = 2^{a-1}(2^a - 1)$. In addition,

$$\mathrm{Irr}^j(B_3) = \begin{cases} \{\chi_{sy,\mu}, \chi_{s,\lambda_i}\colon\ C(sy) = J_\epsilon, \mu = 1, \mathrm{St}\} & \text{if } j = 0,\\ \{\chi_{sy,1}\colon\ C(sy) \simeq \mathbb{Z}_{q^2-1}\} & \text{if } j = a,\\ \{\chi_{sy,1}, \chi_{s,\lambda_3}\colon\ C(sy) = T_\epsilon\} & \text{if } j = 1,\\ \emptyset & \text{otherwise,} \end{cases}$$

where $\lambda_i \in \{\lambda_1, \lambda_2\}$ and λ_3 are unipotent characters of $C_G(s) = L_\epsilon$ such that $\lambda_1(1) = 1$, $\lambda_2(1) = q^3$ and $\lambda_3(1) = q(q + \epsilon)$. Thus $|\mathrm{Irr}^0(B_3)| = 2^{a+1}$, $|\mathrm{Irr}^a(B_3)| = 2^{a-1}$ and $|\mathrm{Irr}^1(B_3)| = 1 + \frac{1}{6}(2^a - 1)(2^a - 2)$. Define

$$\Psi : \mathrm{Irr}^a(B_\epsilon) \longrightarrow \mathrm{Irr}^a(B_3)$$

such that $\Psi(\varphi_{sy,1}) = \chi_{sy,1}$. Then Ψ is a required bijection.

Let $C = C(2)^1$ and $C' = C(2)$. Then $N(C) = N(T_\epsilon)$, $N(C') = \langle T_\epsilon, \rho, \tau\sigma\rangle$ and

$$\mathrm{Blk}\big(N(C)|B\big) = \big\{b(C)\big\}, \qquad \mathrm{Blk}\big(N(C')|B\big) = \big\{b(C')\big\}.$$

If $b \in \mathrm{Blk}(T_\epsilon|b(C'))$, then $b = \mathcal{E}_2(T_\epsilon, (s)) = \{\phi(sy)\colon\ y \in O_3(T_\epsilon)\}$. Suppose

$$y = \mathrm{diag}\{y_1, y_2, y_3\} \in O_2(T_\epsilon).$$

By (3.4), $N(\phi(s))/T_\epsilon \simeq S_3$, and let $\xi_{s,i}$ be the three extensions of $\phi(s)$ in $N(\phi(s))$ such that $\xi_{s,3}(1) = 2$ and $\xi_{s,i}(1) = 1$ for $i = 1, 2$. If $y_i = y_j$ for some $i \neq j \in \{1, 2, 3\}$ and $y \neq 1$, then $C_G(sy) \simeq J_\epsilon$, $N(\phi(sy))/T_\epsilon \simeq \mathbb{Z}_2$ and $\phi(sy)$ has two extensions $\xi_{sy,1}$ and $\xi_{sy,2}$ in $N(\phi(sy))$, and moreover, T_ϵ contains $3(2^a - 1)$ such classes of sy. If $N(\phi(sy)) = T_\epsilon$, then let $\xi_{sy,1} = \phi(sy)$ and T_ϵ has $2^{2a} - 1 - 3(2^a - 1)$ such classes. Let $\eta_{sy,i} = \mathrm{Ind}_{N(\phi(sy))}^{N(C)}(\xi_{sy,i})$. By Clifford theory,

$$\mathrm{Irr}^j\big(b(C)\big) = \begin{cases} \{\eta_{sy,i}, \eta_{s,i}\colon\ C_G(sy) = J_\epsilon,\ i = 1, 2\} & \text{if } j = 0,\\ \{\eta_{sy,1}, \eta_{s,3}\colon\ C_G(sy) = T_\epsilon\} & \text{if } j = 1,\\ \emptyset & \text{otherwise.} \end{cases}$$

Thus $|\mathrm{Irr}^0(b(C))| = 2(2^a - 1) + 2 = 2^{a+1}$ and $|\mathrm{Irr}^1(b(C))| = 1 + \frac{1}{6}(2^a - 1)(2^a - 2)$. Define

$$\Psi : \mathrm{Irr}^j\big(b(C)\big) \longrightarrow \mathrm{Irr}^j(B_3)$$

for $j = 0, 1$ such that $\Psi(\eta_{s,i}) = \chi_{s,\lambda_i}$, $\Psi(\eta_{sy,1}) = \chi_{sy,1}$ and $\Psi(\varphi_{sy,2}) = \chi_{s,\mathrm{St}}$. Then Ψ is a required bijection.

Let $N'(b) = N(b) \cap N(C')$ and $N'(\phi(sy)) = N(\phi(sy)) \cap N(C')$ for $\phi(sy) \in \mathrm{Irr}(b)$. Then $N'(b) = \langle T_\epsilon, \tau\sigma\rangle$ and $N'(\phi(sy)) = N'(b)$ or T_ϵ. In the former case $y = \mathrm{diag}\{y_1, y_1, y_1^{-2}\}$, so that $C_M(sy) = J_\epsilon$ and $\mathrm{Irr}(b)$ has 2^a such characters. Denote by $\zeta_{sy,1}$ and $\zeta_{sy,2}$ the two extensions of $\phi(sy)$ to $N'(\phi(sy))$. Similarly, $\mathrm{Irr}(b)$ has $2^{2a} - 2^a$ characters $\phi(sy)$ such that $N'(\phi(sy)) = T_\epsilon$ and set $\zeta_{sy,1} = \phi(sy)$. Let $\psi_{sy,i} = \mathrm{Ind}_{N(\phi(sy))}^{N(C')}(\zeta_{sy,i})$. Then

$$\mathrm{Irr}^j\big(b(C')\big) = \begin{cases} \{\psi_{sy,i}\colon C_M(sy) = J_\epsilon,\ i = 1,2\} & \text{if } j = 0,\\ \{\psi_{sy,1}\colon C_M(sy) = T_\epsilon\} & \text{if } j = 1,\\ \emptyset & \text{otherwise,}\end{cases}$$

and moreover, $|\mathrm{Irr}^0(b(C'))| = 2^{a+1}$ and $|\mathrm{Irr}^1(b(C'))| = 2^{a-1}(2^a - 1)$. For $j = 0, 1$, define

$$\Psi^{-1}\colon \mathrm{Irr}^j\big(b(C')\big) \longrightarrow \mathrm{Irr}^j(B_\epsilon)$$

such that $\Psi^{-1}(\psi_{sy,1}) = \varphi_{sy,1}$ and $\Psi^{-1}(\psi_{sy,2}) = \varphi_{sy,\mathrm{St}}$.

It follows that Ψ is an $\langle\alpha\rangle$-set isomorphism between $\mathrm{Irr}(b(C)) \cup \mathrm{Irr}(B_\epsilon)$ and $\mathrm{Irr}(b(C')) \cup \mathrm{Irr}(B_3)$. This implies (5A). $\square$

(5B) *Let q be a power of an odd prime p and $B = B_1$ the principal 2-block of $G = G_2(q)$. Then B satisfies Dade's invariant conjecture.*

Proof. Suppose $q \neq 3$ and let $C \in \mathcal{R}^0(G)$. Then $\mathrm{Blk}(N(C)|B_1) = \{b(C)\}$, where $b(C) = B_0(N(C))$.

(1) Set $C = C(1)^1$ and $C' = C(1)$. If $T = T_{-\epsilon}$, then $L = C_G(O_2(T)) = \langle T, \rho\rangle$, $N(C) = N(T)$, and $N(C') = \langle C(C'), \tau\sigma\rangle$. If $b = B_0(L)$ then $E_8 = \langle O_2(T), \rho\rangle$ is a defect group of b, $b^{N(C)} = b(C)$ and $b^{N(C')} = b(C')$. If $b(T) = B_0(T)$, then $\mathrm{Irr}(b(T)) = \phi(O_2(T))$ as $\phi(T) = \phi(O_2(T)) \times \phi(O_{2'}(T))$. We can view b as the principal block of $E_8 = \langle O_2(T), \rho\rangle$. Thus $b(C)$ and $b(C')$ are the principal blocks of $\langle O_2(T), \rho, \sigma, \tau\rangle$ and $\langle O_2(T), \rho, \tau\sigma\rangle$, respectively. As shown in the proof of (3D) (3), (5.1) holds.

(2) Let $C = C(2)^1$ or $C(2)_1$ according as $2^a \neq q - \epsilon$ or $2^a = q - \epsilon$, and let $C' = C(2)$. If $T = T_\epsilon$, then $N(C) = N(T)$ and $N(C') = \langle T, \rho, \tau\sigma\rangle$. Now $\phi(y) \in \Omega_1(\mathrm{Irr}(b))$ if and only if $y \in \Omega_1(O_2(T))$, so $D_{12} = N(T)/T$ and β permutes $\Omega_1(\mathrm{Irr}(b))$, where $b = B_0(T)$. Let $\mathcal{X}_1(C)$ and $\mathcal{X}_1(C')$ be the subsets of $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(b(C'))$, respectively consisting of characters covering $\Omega_1(\mathrm{Irr}(b))$. A proof similar to that of (3D) (3) shows that there is a required bijection Ψ between $\mathcal{X}_1(C')$ and $\mathcal{X}_1(C)$.

Let $\mathcal{Z}(\tau\sigma)$ be characters of $\mathrm{Irr}(b)\backslash\Omega_1(\mathrm{Irr}(b))$ stabilized by $\tau\sigma$. By (3.4), $|\mathcal{Z}(\tau\sigma)| = 2^a - 2$ and ρ permutes $\mathcal{Z}(\tau\sigma)$. Similarly, $|\mathcal{Z}(\rho\tau\sigma)| = 2^a - 2$ and ρ permutes $\mathcal{Z}(\rho\tau\sigma)$. Moreover, if $p = 3$, then β' interchanges $\mathcal{Z}(\tau\sigma)$ and $\mathcal{Z}(\rho\tau\sigma)$. Thus $\mathrm{Irr}(b)$ has $2^{2a} - 3(2^a - 2) - 3(2^a - 2) - 4$ characters $\phi(y)$ such that $N(\phi(y)) = T$. Let $\xi_{y,1}$ and $\xi_{y,2}$ be the two extensions of $\phi(y) \in \mathcal{Z}(\tau\sigma) \cup \mathcal{Z}(\rho\tau\sigma)$ to $N(\phi(y))$, and

$$\mathcal{X}_2(C) = \big\{\varphi_{y,i} = \mathrm{Ind}_{N(\phi(y))}^{N(C)}(\xi_{y,i})\colon \phi(y) \in \mathcal{Z}(\tau\sigma) \cup \mathcal{Z}(\rho\tau\sigma),\ i = 1,2\big\}.$$

Then $|\mathcal{X}_2(C)| = 2(2^a - 2)$ and

$$\mathrm{Irr}^2\big(b(C)\big) = \big\{\varphi_{y,1} = \mathrm{Ind}_T^{N(C)}\big(\phi(y)\big)\big\},$$

where $\phi(y) \in \operatorname{Irr}(b)\backslash(\Omega_1(\operatorname{Irr}(b)) \cup \mathcal{Z}(\tau\sigma) \cup \mathcal{Z}(\rho\tau\sigma))$. So $|\operatorname{Irr}^2(b(C))| = \frac{1}{12}(2^{2a} - 3 \times 2^{a+1} + 8)$ and if $\varphi_{y,1} \in \operatorname{Irr}^2(b(C))$, then $C(y) = T$.

Similarly, let $\zeta_{y,1}$ and $\zeta_{y,2}$ be the two extensions of $\phi(y) \in \mathcal{Z}(\tau\sigma) \cup \mathcal{Z}(\rho\tau\sigma)$ in $N'(\phi(y)) = N(\phi(y)) \cap N(C')$, and

$$\mathcal{X}_2(C') = \left\{\psi_{y,i} = \operatorname{Ind}_{N(\phi(y))}^{N(C')}(\zeta_{y,i})\colon\ \phi(y) \in \mathcal{Z}(\tau\sigma) \cup \mathcal{Z}(\rho\tau\sigma),\ i = 1, 2\right\}.$$

Then $|\mathcal{X}_2(C')| = 2(2^a - 2)$ and

$$\operatorname{Irr}^2\big(b(C')\big) = \left\{\psi_{y,1} = \operatorname{Ind}_T^{N(C')}\big(\phi(y)\big)\right\},$$

where $\phi(y) \in \operatorname{Irr}(b)\backslash(\Omega_1(\operatorname{Irr}(b)) \cup \mathcal{Z}(\tau\sigma) \cup \mathcal{Z}(\rho\tau\sigma))$. So $|\operatorname{Irr}^2(b(C'))| = \frac{1}{4}(2^{2a} - 2^{a+1})$ and if $\psi_{y,1} \in \operatorname{Irr}^2(b(C'))$, then we may suppose $y = \operatorname{diag}\{y_1, y_2, y_3\} \in T_\epsilon$ such that $y_2 \notin \{y_1, y_1^{-1}\}$ and $y_1 y_2 y_3 = 1$. Define

$$\Psi : \mathcal{X}_2(C') \longrightarrow \mathcal{X}_2(C)$$

such that $\Psi(\psi_{y,i}) = \varphi_{y,i}$. Then Ψ is a $\langle\beta\rangle$-set isomorphism of $\operatorname{Irr}^0(b(C')) \cup \operatorname{Irr}^1(b(C'))$ and $\operatorname{Irr}^0(b(C)) \cup \operatorname{Irr}^1(b(C))$. Moreover, if $\psi_{y,1} \in \operatorname{Irr}^2(b(C'))$ or $\varphi_{y,1} \in \operatorname{Irr}^2(b(C))$, then $C_G(y) = T$, $\psi_{y,1}^\beta = \psi_{y^\beta,1}$ and $\varphi_{y,1}^\beta = \varphi_{y^\beta,1}$.

(3) Let $H = N(C(1)_1) = \mathrm{SO}(U_4)$ and $b_H = B_0(H)$, so that $b_H = \mathcal{E}_2(H, (1))$. Let $\mathcal{Y}(Q) = \mathcal{Y}(b_H, Q)$ be defined as (4.2), and $\mathcal{Y}_0 = \{\chi_1, \chi_2, \varphi_1, \varphi_2\}$, where $\chi_i(1) = \frac{1}{2}(q-\epsilon)^2$ and $\varphi_i(1) = \frac{1}{2}(q+\epsilon)^2$ which are defined in the proof (5) of [3, (4C)]. As shown in the proof (5) of [3, (4C)]

$$|\mathcal{Y}| = \begin{cases} 2^{a-1} & \text{if } \mathcal{Y} = \mathcal{Y}(\mathbb{Z}_{q^2-1}), \\ 2(2^a - 2) & \text{if } \mathcal{Y} = \mathcal{Y}(J_\epsilon), \\ \frac{1}{4}(2^{2a} - 2^{a+1}) & \text{if } \mathcal{Y} = \mathcal{Y}(T_\epsilon), \\ 8 & \text{if } \mathcal{Y} = \mathcal{Y}(Z(H)), \\ 4 & \text{if } \mathcal{Y} = \mathcal{Y}_0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Moreover, $\mathcal{Y}(H) = \{\varphi_{1i}, \varphi_{2i}\colon\ 1 \leqslant i \leqslant 4\}$, where $\varphi_{1i}(1) = \varphi_{2i}(1)$, $\varphi_{11}(1) = 1$, $\varphi_{12}(1) = q^2$, and $\varphi_{13}(1) = \varphi_{14}(1) = q$. In addition, if $\chi \in \mathcal{Y}(Q)$ with height $\mathrm{h}_Q = \mathrm{h}(\chi)$, then

$$(\mathrm{h}_{\mathbb{Z}_{q^2-1}}, \mathrm{h}_{J_\epsilon}, \mathrm{h}_{T_\epsilon}, \mathrm{h}_H) = (a+1, 1, 2, 0)$$

and $\mathrm{h}(\chi_i) = 2a - 1$, $\mathrm{h}(\varphi_i) = 1$ for $i = 1, 2$.

In the notation of (3.5), $H = \langle M_1 \circ M_2, g\rangle$, where $M_1 \simeq M_2 \simeq \mathrm{SL}_2(q)$. As shown in the proof of [3, (4C)] $\chi_1 = \chi' \otimes \chi'' + \chi'' \otimes \chi' \in \mathcal{Y}_0$, where $\chi', \chi'' \in \operatorname{Irr}(\mathrm{SL}_2(q))$ with degree $\frac{1}{2}(q-\epsilon)$. By (3B), $M_i^\alpha = M_i$ and so $\chi_1^\alpha = \chi_1$. If $p = 3$, then $M_1^{\beta'} = M_2$ and also $\chi_1^{\beta'} = \chi_1$. Similarly, β' stabilizes all the other characters of $\mathcal{Y}_0$.

Thus β' stabilizes each character of $\operatorname{Irr}^{2a-1}(b_H) = \{\chi_1, \chi_2\}$. Since $\operatorname{Irr}^{2a-1}(B_1) = \{X_{17}, X_{18}\}$ or $\{X_{15}, X_{16}\}$ according as $\epsilon = +$ or $-$, it follows by (4A) that β stabilizes each character of $\operatorname{Irr}^{2a-1}(B_1)$. Thus there exists a required bijection Ψ of $\operatorname{Irr}^{2a-1}(b_H)$ onto $\operatorname{Irr}^{2a-1}(B_1)$.

Each character of degree q in $\operatorname{Irr}^0(b_H) = \mathcal{Y}(H)$ is an extension of a central product of the trivial character 1 and the Steinberg character St of $\mathrm{SL}_2(q)$. It follows from (3B) that α stabilizes each character of $\mathrm{Irr}^0(b_H)$. If $p=3$, then $M_1^{\beta'}=M_2$, so that $(1\otimes \mathrm{St})^{\beta'}=\mathrm{St}\otimes 1$ and β' stabilizes no characters of degree q, so $\varphi_{13}^{\beta'}=\varphi_{14}$ and $\varphi_{23}^{\beta'}=\varphi_{24}$. But $\mathrm{Irr}^0(B_1)=\{X_{1i},X_{2i}\colon 1\leqslant i\leqslant 4\}$, so the map $\Psi(\varphi_{ij})=X_{ij}$ is a required bijection of $\mathrm{Irr}^0(b_H)$ onto $\mathrm{Irr}^0(B_1)$.

Similarly, $\mathrm{Irr}^1(b_H)=\{\varphi_1,\varphi_2\}\cup\mathcal{Y}(b_H,J_\epsilon)$ and $\mathrm{Irr}^1(B_1)=\{X_{15},X_{16}\}\cup\mathcal{Y}(B_1,J_\epsilon)$ or $\{X_{17},X_{18}\}\cup\mathcal{Y}(B_1,J_\epsilon)$ according as $\epsilon=+$ or $-$. Define $\Psi(\varphi_1)=X_{15}$ or X_{17}, $\Psi(\varphi_2)=X_{16}$ or X_{18} according as $\epsilon=+$ or $-$, and $\Psi(\varphi_{y,\mu})=\chi_{y,\mu}$, where $\varphi_{y,\mu}\in\mathcal{Y}(b_H,J_\epsilon)$ with $\mu=1,\mathrm{St}$. Thus Ψ is a required bijection of $\mathrm{Irr}^1(b_H)$ onto $\mathrm{Irr}^1(B_1)$.

Since $\mathrm{Irr}^{a+1}(b_H)=\mathcal{Y}(b_H,\mathbb{Z}_{q^2-1})$ and $\mathrm{Irr}^{a+1}(B_1)=\mathcal{Y}(B_1,\mathbb{Z}_{q^2-1})$, it follows that $\Psi(\varphi_{y,1})=\chi_{y,1}$ is a required bijection of $\mathrm{Irr}^{a+1}(b_H)$ onto $\mathrm{Irr}^{a+1}(B)$.

Finally, $\mathrm{Irr}^2(b_H)=\mathcal{Y}(b_H,T_\epsilon)$, $|\mathrm{Irr}^2(b_H)|=\frac{1}{4}(2^{2a}-2^{a+1})$ and if $\zeta_{y,1}\in\mathrm{Irr}^2(b_H)$, we may suppose $y=\mathrm{diag}\{y_1,y_2,y_3\}\in T_\epsilon$ such that $y_2\neq\{y_1,y_1^{-1}\}$. By [21], $\mathrm{Irr}^2(B_1)=\mathcal{Y}(B_1,T_\epsilon)$, $|\mathrm{Irr}^2(B_1)|=\frac{1}{12}(2^a-4)(2^a-2)$ and if $\chi_{y,1}\in\mathrm{Irr}^2(B_1)$, then $C_G(y)=T_\epsilon$. Define

$$\Psi:\mathrm{Irr}^2\big(b(C)\big)\longrightarrow\mathrm{Irr}^2(B_1)$$

such that $\Psi(\varphi_{y,1})=\chi_{y,1}$ for $\varphi_{y,1}\in\mathrm{Irr}^2(b(C))$ and also define $\Psi:\mathrm{Irr}^2(b(C'))\to\mathrm{Irr}^2(b_H)$ such that $\Psi(\psi_{y,1})=\zeta_{y,1}$, where $\mathrm{Irr}^2(b(C))$ and $\mathrm{Irr}^2(b(C'))$ are given by (2). Then Ψ is a required bijection. It follows by (2) and (3) that

$$\Psi:\mathrm{Irr}\big(b(C')\big)\cup\mathrm{Irr}(B_1)\longrightarrow\mathrm{Irr}(b_H)\cup\mathrm{Irr}\big(b(C)\big)$$

is a $\langle\beta\rangle$-set isomorphism. Thus

$$\mathrm{k}(G,B_1,d,u)+\mathrm{k}\big(N(C'),B_1,d,u\big)=\mathrm{k}(H,B_1,d,u)+\mathrm{k}\big(N(C),B_1,d,u\big).$$

(4) Suppose $C\in\{C(3)_1,C(3),C(4)_2,C(4)\}$, so that $N(C)=N_{G_2(p)}(C)$. If $p\neq 3$, then $N_A(C)=N(C)\times\langle\alpha\rangle$ and we can apply the proofs (3) and (4) of [3, (4C)], so that (5B) holds. Suppose $p=3$, then $N_A(C)=N_{G_2(3):2}(C)\times\langle\alpha\rangle$, where $G_2(3){:}2=\mathrm{Aut}(G_2(3))$. We may suppose $q=p=3$, so that $A=G_2(3){:}2$. The character tables of $N(C)$ and $N_A(C)$ can be calculated directly using the structure given in Section 3 (cf. the proof (3) of (3D)), or calculated easily by computer. We omit the calculations here.

Set $\mathrm{k}(C,d,u)=\mathrm{k}(N(C),B_1,d,u)$. Then its values are given in Table 1.

It follows that

$$\sum_{C\in\{C(3)_1,C(3),C(4)_2,C(4)\}}(-1)^{|C|}\mathrm{k}\big(N(C),B_1,d,u\big)=0,$$

which implies (5B).

Table 1
The values $\mathrm{k}(N(C),B_1,d,u)$

Defect d	6	6	5	5	4	3	otherwise
Value u	2	1	2	1	1	2	otherwise
$\mathrm{k}(C(3)_1,d,u)$	4	4	2	0	0	1	0
$\mathrm{k}(C(3),d,u)$	4	4	2	0	2	1	0
$\mathrm{k}(C(4)_2,d,u)$	4	4	2	4	0	0	0
$\mathrm{k}(C(4),d,u)$	4	4	2	4	2	0	0

Table 2
The values $\mathrm{k}(N(C), B_1, d, u)$ when $q = 3$

Defect d	6	6	5	5	4	3	3	otherwise
Value u	2	1	2	1	1	2	1	otherwise
$\mathrm{k}(C(3)^1, d, u)$	4	4	2	4	0	2	2	0
$\mathrm{k}(C(3)_0, d, u)$	4	4	2	4	2	2	2	0

(5) Finally, suppose $q = p = 3$ and $C \in \{C(3)_0, C(3)_1, C(3)^1, C(3)\}$. If $C = C(3)_1$ or $C(3)$, then the values $\mathrm{k}(C, d, u)$ are given by Table 1. The values $\mathrm{k}(C(3)_0, d, u)$ and $\mathrm{k}(C(3)^1, d, u)$ are given in Table 2.

It follows that

$$\sum_{C \in \{C(3)_0, C(3)_1, C(3)^1, C(3)\}} (-1)^{|C|} \mathrm{k}\big(N(C), B_1, d, u\big) = 0,$$

which implies (5B). $\square$

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